PLAYING WITH EIGENVALUES-PUTNAM SEMINAR 2016

1. Determine the number of possible values for the determinant of A, given that A is a $n \times n$ matrix with real entries such that $A^3 - A^2 - 3A + 2I = 0$, where I is the identity and 0 is the all-zero matrix.

2. Consider matrices $A_1, A_2, ..., A_m \in M_n(\mathbb{R})$ not all nilpotent. Prove that there is an integer number $k > 0$ such that $A_1^k + A_2^k + ... + A_m^k \neq O_n$

3. Let $A, B \in M_2(\mathbb{C})$ such that : $A^2 + B^2 = 2AB$.
   a) Prove that : $AB = BA$.
   b) Prove that : $\operatorname{tr}(A) = \operatorname{tr}(B)$.

4. Let A, B be matrices of dimension 2010×2010 which commute and have real entries, such that $A^{2010} = B^{2010} = I$, where I is the identity matrix. Prove that if $\operatorname{tr}(AB) = 2010$, then $\operatorname{tr}(A) = \operatorname{tr}(B)$.

5. Let A and B be real symmetric matrixes with all eigenvalues strictly greater than 1. Let λ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

6. Does there exist a real 3×3 matrix A such that $\operatorname{tr}(A) = 0$ and $A^2 + A^t = I$? ($\operatorname{tr}(A)$ denotes the trace of A, A^t the transpose of A, and I is the identity matrix.)

7. let $A, B \in M_n(\mathbb{C})$. If $A(AB - BA) = (AB - BA)A$ prove that $AB - BA$ is nilpotent.

8. Let A be a real symmetric matrix and $B \in M_n(\mathbb{C})$ such that $AB + BA = 0$. Prove that $AB = 0$.

9. Let $A \in \mathrm{Sl}_3(\mathbb{Z})$ of finite order. Find all possible values of $\operatorname{tr}(A)$.

10. Let A and B be two complex matrices. Prove that the following conditions are equivalent:
    a) For any $M \in M_n(\mathbb{C})$ the characteristic polynomials of AM and $AM + B$ are the same
    b) B is nilpotent and $BA = O_n$.

11. $A, B \in M_2(\mathbb{C})$ with $\det(A) = 1$,—$\operatorname{tr}(A)| \neq 2$,$\det(B) = 1$, $|\operatorname{tr}(B)| \neq 2$ and suppose also A, B do not have common eigenvectors. Given that there exist $(n_1, ..., n_k, m_1, ..., m_k) \in \mathbb{Z}$ such that $A^{n_1} B^{m_1} ... A^{n_k} B^{m_k} = I_2$ prove that $A^{-n_1} B^{-m_1} ... A^{-n_k} B^{-m_k} = I_2$

12. Let $A, B \in M_2(\mathbb{C})$ with $\exp(A) = \exp(B)$. Suppose for any eigenvalue a of A and b of B, $a - b \notin 2\pi i\mathbb{Z}$. Then $A = B$.

13. Let A, B be square matrices a) of size 2016×2016; b) of size 2017×2017. Do there necessarily exist real numbers a, b such that $a^2 + b^2 \neq 0$ and the matrix $aA + bB$ is singular?

14. Let a square matrix P be neither zero nor unit and such that $P^2 = P$. Does there always exist such a matrix Q that $Q^2 = Q$, $PQ = QPQ$ but $QP \neq PQ$?