Putnam Club Week Nine – 13 November 2012

Grab Bag

1. Let h and k be positive integers. Prove that for every $\epsilon > 0$, there are positive integers m and n such that

$$\epsilon < |h\sqrt{m} - k\sqrt{n}| < 2\epsilon.$$

(Putnam 2011)

2. What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.) (Putnam 2008)

3. Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

(Putnam 2006)

4. Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N? (Putnam 2004)

5. Prove that for $n \geq 2$,

$$\overbrace{2^{2^{\cdots^{2}}}}^{n \text{ terms}} \equiv \overbrace{2^{2^{\cdots^{2}}}}^{n-1 \text{ terms}} \pmod{n}.$$

(Putnam 1997)

6. Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite? (Putnam 1989)