10/12/11 – Polynomials (Hardcore)

Note: Some of these problems require use of complex numbers.

1. Find the minimum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$

for $x>0$. (Putnam 1998)

2. Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x),\ldots,f_k(x)$ such that

$$p(x)=\sum_{j=1}^{k}(f_j(x))^2.$$

(Putnam 1999)

3. Let $P(x)$ be a polynomial of degree n such that $P(x)=Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots. (Putnam 1999)

4. Find all pairs of real numbers (x,y) satisfying the system of equations

$$\frac{1}{x}+\frac{1}{2y}=(x^2+3y^2)(3x^2+y^2)$$
$$\frac{1}{x}-\frac{1}{2y}=2(y^4-x^4).$$

(Putnam 2001)

5. Do there exist polynomials $a(x),b(x),c(y),d(y)$ such that

$$1+xy+x^2y^2=a(x)c(y)+b(x)d(y)$$

holds identically? (Putnam 2003)

6. Let

$$\begin{aligned}f(z)&=az^4+bz^3+cz^2+dz+e\\&=a(z-r_1)(z-r_2)(z-r_3)(z-r_4)\end{aligned}$$

where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number. (Putnam 2003)

7. Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

(Putnam 2004)

8. Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$

and $\deg P > \deg Q$. (Putnam 2007)

9. Show that the solutions to the equation $x^5 + 6x^4 + 14x^3 + 31x^2 + 41x + 3 = 0$ are not all negative real numbers.

10. Find all polynomials $p(x)$ such that $p(nm + 1) = p(n)p(m) + 1$ for all integers n, m.