Fall 2021

Wednesday, November 22nd, 2021

Last time we started working on the following 5 problems. For some of them we found solutions, but for some we did not. Let's pick up from where we left ! I am adding more problems to this working sheet, hopping they will be a good exercise for the coming up exam!

1. (Putnam B3 2005) Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

Proof. Substitute a/x for x in the given equation:

$$f'(x) = \frac{a}{xf(a/x)}.$$

Differentiate:

$$f''(x) = -\frac{a}{x^2 f(a/x)} + \frac{a^2 f'(a/x)}{x^3 f(a/x)^2}.$$

Now substitute to eliminate evaluations at a/x:

$$f''(x) = -\frac{f'(x)}{x} + \frac{f'(x)^2}{f(x)}.$$

Clear denominators:

$$xf(x)f''(x) + f(x)f'(x) = xf'(x)^2.$$

Divide through by $f(x)^2$ and rearrange:

$$0 = \frac{f'(x)}{f(x)} + \frac{xf''(x)}{f(x)} - \frac{xf'(x)^2}{f(x)^2}.$$

The right side is the derivative of $xf'(x)/f(x)$, so that quantity is constant. That is, for some d,

$$\frac{f'(x)}{f(x)} = \frac{d}{x}.$$

Integrating yields $f(x) = cx^d$, as desired.

□

2. Suppose that f is a function on the interval $[1, 3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,dx$ be?

Solution: In all solutions, we assume that the function f is integrable.

Proof. Let $g(x)$ be 1 for $1 \le x \le 2$ and -1 for $2 < x \le 3$, and define $h(x) = g(x) - f(x)$. Then $\int_1^3 h(x)\,dx = 0$ and $h(x) \ge 0$ for $1 \le x \le 2$, $h(x) \le 0$ for $2 < x \le 3$. Now

$$\int_1^3 \frac{h(x)}{x}\,dx = \int_1^2 \frac{|h(x)|}{x}\,dx - \int_2^3 \frac{|h(x)|}{x}\,dx$$
$$\ge \int_1^2 \frac{|h(x)|}{2}\,dx - \int_2^3 \frac{|h(x)|}{2}\,dx = 0,$$

and thus $\int_1^3 \frac{f(x)}{x}\,dx \le \int_1^3 \frac{g(x)}{x}\,dx = 2\log 2 - \log 3 = \log \frac{4}{3}$. Since $g(x)$ achieves the upper bound, the answer is $\log \frac{4}{3}$.

□

3. (Putnam and beyound exemplu pag 132) Let $f : [0,1] \to \mathbb{R}$ be a continuous function with the property that

$$\int_0^1 f(x)\,dx = \frac{\pi}{4}.$$

Prove that there exists $x_0 \in (0,1)$ such that

$$\frac{1}{1+x_0} < f(x_0) < \frac{1}{2x_0}.$$

Proof. to come! □

4. Let $f : [0,\infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that

$$\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\,dx$$

diverges.

Proof. **Solution: First solution.** Note that the hypotheses on f imply that $f(x) > 0$ for all $x \in [0,+\infty)$, so the integrand is a continuous function of f and the integral makes sense. Rewrite the integral as

$$\int_0^\infty \left(1 - \frac{f(x+1)}{f(x)}\right)\,dx,$$

and suppose by way of contradiction that it converges to a finite limit L. For $n \ge 0$, define the Lebesgue measurable set

$$I_n = \{x \in [0,1] : 1 - \frac{f(x+n+1)}{f(x+n)} \le 1/2\}.$$

Then $L \geq \sum_{n=0}^{\infty} \frac{1}{2}(1-\mu(I_n))$, so the latter sum converges. In particular, there exists a nonnegative integer N for which $\sum_{n=N}^{\infty}(1-\mu(I_n)) < 1$; the intersection

$$I = \bigcup_{n=N}^{\infty} I_n = [0,1] - \bigcap_{n=N}^{\infty}([0,1] - I_n)$$

then has positive Lebesgue measure.

By Taylor's theorem with remainder, for $t \in [0,1/2]$,

$$\begin{aligned} -\log(1-t) &\leq t + t^2 \sup_{t\in[0,1/2]} \left\{ \frac{1}{(1-t)^2} \right\} \\ &= t + \frac{4}{3}t^2 \leq \frac{5}{3}t. \end{aligned}$$

For each nonnegative integer $n \geq N$, we then have

$$\begin{aligned} L &\geq \int_N^n \left(1 - \frac{f(x+1)}{f(x)}\right) dx \\ &= \sum_{i=N}^{n-1} \int_0^1 \left(1 - \frac{f(x+i+1)}{f(x+i)}\right) dx \\ &\geq \sum_{i=N}^{n-1} \int_I \left(1 - \frac{f(x+i+1)}{f(x+i)}\right) dx \\ &\geq \frac{3}{5} \sum_{i=N}^{n-1} \int_I \log \frac{f(x+i)}{f(x+i+1)}\, dx \\ &= \frac{3}{5} \int_I \left(\sum_{i=N}^{n-1} \log \frac{f(x+i)}{f(x+i+1)} \right) dx \\ &= \frac{3}{5} \int_I \log \frac{f(x+N)}{f(x+n)}\, dx. \end{aligned}$$

For each $x \in I$, $\log f(x+N)/f(x+n)$ is a strictly increasing unbounded function of n. By the monotone convergence theorem, the integral $\int_I \log(f(x+N)/f(x+n))\,dx$ grows without bound as $n \to +\infty$, a contradiction. Thus the original integral diverges, as desired.

Remark. This solution is motivated by the commonly-used fact that an infinite product $(1+x_1)(1+x_2)\cdots$ converges absolutely if and only if the sum $x_1+x_2+\cdots$ converges absolutely. The additional measure-theoretic argument at the beginning is needed because one cannot bound $-\log(1-t)$ by a fixed multiple of t uniformly for all $t \in [0,1)$.

Greg Martin suggests a variant solution that avoids use of Lebesgue measure. Note first that if $f(y) > 2f(y+1)$, then either $f(y) > \sqrt{2}f(y+1/2)$ or $f(y+1/2) > \sqrt{2}f(y+1)$, and in either case we deduce that

$$\int_{y-1/2}^{y+1/2} \frac{f(x)-f(x+1)}{f(x)}\, dx > \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) > \frac{1}{7}.$$

If there exist arbitrarily large values of y for which $f(y) > 2f(y+1)$, we deduce that the original integral is greater than any multiple of $1/7$, and so diverges. Otherwise, for x large we may argue that

$$\frac{f(x)-f(x+1)}{f(x)} > \frac{3}{5}\log\frac{f(x)}{f(x+1)}$$

as in the above solution, and again get divergence using a telescoping sum.

Second solution. (Communicated by Paul Allen.) Let $b > a$ be nonnegative integers. Then

$$\begin{aligned}
\int_a^b \frac{f(x)-f(x+1)}{f(x)}dx &= \sum_{k=a}^{b-1}\int_0^1 \frac{f(x+k)-f(x+k+1)}{f(x+k)}dx \\
&= \int_0^1 \sum_{k=a}^{b-1}\frac{f(x+k)-f(x+k+1)}{f(x+k)}dx \\
&\geq \int_0^1 \sum_{k=a}^{b-1}\frac{f(x+k)-f(x+k+1)}{f(x+a)}dx \\
&= \int_0^1 \frac{f(x+a)-f(x+b)}{f(x+a)}dx.
\end{aligned}$$

Now since $f(x) \to 0$, given a, we can choose an integer $l(a) > a$ for which $f(l(a)) < f(a+1)/2$; then $\frac{f(x+a)-f(x+l(a))}{f(x+a)} \geq 1 - \frac{f(l(a))}{f(a+1)} > 1/2$ for all $x \in [0,1]$. Thus if we define a sequence of integers a_n by $a_0 = 0$, $a_{n+1} = l(a_n)$, then

$$\begin{aligned}
\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}dx &= \sum_{n=0}^\infty \int_{a_n}^{a_{n+1}} \frac{f(x)-f(x+1)}{f(x)}dx \\
&> \sum_{n=0}^\infty \int_0^1 (1/2)dx,
\end{aligned}$$

and the final sum clearly diverges.

Third solution. (By Joshua Rosenberg, communicated by Catalin Zara.) If the original integral converges, then on one hand the integrand $(f(x)-f(x+1))/f(x) = 1 - f(x+1)/f(x)$ cannot tend to 1 as $x \to \infty$. On the other hand, for any $a \geq 0$,

$$\begin{aligned}
0 &< \frac{f(a+1)}{f(a)} \\
&< \frac{1}{f(a)}\int_a^{a+1} f(x)\,dx \\
&= \frac{1}{f(a)}\int_a^\infty (f(x)-f(x+1))\,dx \\
&\leq \int_a^\infty \frac{f(x)-f(x+1)}{f(x)}\,dx,
\end{aligned}$$

and the last expression tends to 0 as $a \to \infty$. Hence by the squeeze theorem, $f(a+1)/f(a) \to 0$ as $a \to \infty$, a contradiction. □

5. Let $f : [0,1] \to \mathbb{R}$ be a function for which there exists a constant $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in [0,1]$. Suppose also that for each rational number $r \in [0,1]$, there exist integers a and b such that $f(r) = a + br$. Prove that there exist finitely many intervals $I_1, \dots, I_n$ such that f is a linear function on each I_i and $[0,1] = \bigcup_{i=1}^n I_i$.

Proof. Let us say that a linear function g on an interval is *integral* if it has the form $g(x) = a + bx$ for some $a, b \in \mathbb{Z}$, and that a piecewise linear function is *integral* if on every interval where it is linear, it is also integral.

For each positive integer n, define the n-th *Farey sequence* F_n as the sequence of rational numbers in $[0,1]$ with denominators at most n. It is easily shown by induction on n that any two consecutive elements $\frac{r}{s}, \frac{r'}{s'}$ of F_n, written in lowest terms, satisfy $\gcd(s, s') = 1$, $s + s' > n$, and $r's - rs' = 1$. Namely, this is obvious for $n = 1$ because $F_1 = \frac{0}{1}, \frac{1}{1}$. To deduce the claim for F_n from the claim for F_{n-1}, let $\frac{r}{s}, \frac{r'}{s'}$ be consecutive elements of F_{n-1}. If $s + s' = n$, then for $m = r + r'$ we have $\frac{r}{s} < \frac{m}{n} < \frac{r'}{s'}$ and the pairs $\frac{r}{s}, \frac{m}{n}$ and $\frac{m}{n}, \frac{r'}{s'}$ satisfy the desired conditions. Conversely, if $s + s' > n$, then we cannot have $\frac{r}{s} < \frac{m}{n} < \frac{r'}{s'}$ for $a \in \mathbb{Z}$, as this yields the contradiction

$$n = (ms - nr)s' + (r'n - ms') \geq s + s' > n;$$

hence $\frac{r}{s}, \frac{r'}{s'}$ remain consecutive in F_n.

Let $f_n : [0,1] \to \mathbb{R}$ be the piecewise linear function which agrees with f at each element of F_n and is linear between any two consecutive elements of F_n. Between any two consecutive elements $\frac{r}{s}, \frac{r'}{s'}$ of F_n, f_n coincides with some linear function $a + bx$. Since $sf(\frac{r}{s}), s'f(\frac{r'}{s'}) \in \mathbb{Z}$, we deduce first that

$$b = ss'(f(\frac{r'}{s'}) - f(\frac{r}{s}))$$

is an integer of absolute value at most K, and second that both $as = sf(\frac{r}{s}) - br$ and $as' = s'f(\frac{r'}{s'}) - br'$ are integral. It follows that f_n is integral.

We now check that if $n > 2K$, then $f_n = f_{n-1}$. For this, it suffices to check that for any consecutive elements $\frac{r}{s}, \frac{m}{n}, \frac{r'}{s'}$ in F_n, the linear function $a_0 + b_0 x$ matching f_{n-1} from $\frac{r}{s}$ to $\frac{r'}{s'}$ has the property that $f(\frac{m}{n}) = a_0 + b_0 \frac{m}{n}$. Define the integer $t = nf(\frac{m}{n}) - a_0 n - b_0 m$. We then compute that the slope of f_n from $\frac{r}{s}$ to $\frac{m}{n}$ is $b_0 + st$, while the slope of f_n from $\frac{m}{n}$ to $\frac{r'}{s'}$ is at most $b_0 - s't$. In order to have $|b_0 + st|, |b_0 - s't| \leq K$, we must have $(s + s')|t| \leq 2K$; since $s + s' = n > 2K$, this is only possible if $t = 0$. Hence $f_n = f_{n-1}$, as claimed.

It follows that for any $n > 2K$, we must have $f_n = f_{n+1} = \cdots$. Since the condition on f and K implies that f is continuous, we must also have $f_n = f$, completing the proof.

□

Remark: The condition on f and K is called *Lipschitz continuity.*

Remark: An alternate approach is to prove that for each $x \in [0, 1)$, there exists $\epsilon \in (0, 1 - x)$ such that the restriction of f to $[x, x + \epsilon)$ is linear; one may then deduce the claim using the compactness of $[0, 1]$. In this approach, the role of the Farey sequence may also be played by the convergents of the continued fraction of x (at least in the case where x is irrational).

Remark: This problem and solution are due to one of us (Kedlaya). Some related results can be proved with the Lipschitz continuity condition replaced by suitable convexity conditions. See for example: Kiran S. Kedlaya and Philip Tynan, Detecting integral polyhedral functions, *Confluentes Mathematici* **1** (2009), 87–109. Such results arise in the theory of p-adic differential equations; see for example: Kiran S. Kedlaya and Liang Xiao, Differential modules on p-adic polyannuli, *J. Inst. Math. Jusssieu* **9** (2010), 155–201 (errata, *ibid.*, 669–671).

6. ([B5] 2009) Let $f : (1, \infty) \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - f(x)^2}{x^2(f(x)^2 + 1)} \qquad \text{for all } x > 1.$$

Prove that $\lim_{x\to\infty} f(x) = \infty$.

Proof. **First solution.** If $f(x) \geq x$ for all $x > 1$, then the desired conclusion clearly holds. We may thus assume hereafter that there exists $x_0 > 1$ for which $f(x_0) < x_0$.

Rewrite the original differential equation as

$$f'(x) = 1 - \frac{x^2 + 1}{x^2} \frac{f(x)^2}{1 + f(x)^2}.$$

Put $c_0 = \min\{0, f(x_0) - 1/x_0\}$. For all $x \geq x_0$, we have $f'(x) > -1/x^2$ and so

$$f(x) \geq f(x_0) - \int_{x_0}^{x} dt/t^2 > c_0.$$

In the other direction, we claim that $f(x) < x$ for all $x \geq x_0$. To see this, suppose the contrary; then by continuity, there is a least $x \geq x_0$ for which $f(x) \geq x$, and this least value satisfies $f(x) = x$. However, this forces $f'(x) = 0 < 1$ and so $f(x - \epsilon) > x - \epsilon$ for $\epsilon > 0$ small, contradicting the choice of x.

Put $x_1 = \max\{x_0, -c_0\}$. For $x \geq x_1$, we have $|f(x)| < x$ and so $f'(x) > 0$. In particular, the limit $\lim_{x\to+\infty} f(x) = L$ exists.

Suppose that $L < +\infty$; then $\lim_{x\to+\infty} f'(x) = 1/(1 + L^2) > 0$. Hence for any sufficiently small $\epsilon > 0$, we can choose $x_2 \geq x_1$ so that $f'(x) \geq \epsilon$ for $x \geq x_2$. But then $f(x) \geq f(x_2) + \epsilon(x - x_2)$, which contradicts $L < +\infty$. Hence $L = +\infty$, as desired.

Variant. (by Leonid Shteyman) One obtains a similar argument by writing

$$f'(x) = \frac{1}{1+f(x)^2} - \frac{f(x)^2}{x^2(1+f(x)^2)},$$

so that

$$-\frac{1}{x^2} \leq f'(x) - \frac{1}{1+f(x)^2} \leq 0.$$

Hence $f'(x) - 1/(1+f(x)^2)$ tends to 0 as $x \to +\infty$, so $f(x)$ is bounded below, and tends to $+\infty$ if and only if the improper integral $\int dx/(1+f(x)^2)$ diverges. However, if the integral were to converge, then as $x \to +\infty$ we would have $1/(1+f(x)^2) \to 0$; however, since f is bounded below, this again forces $f(x) \to +\infty$.

Second solution. (by Catalin Zara) The function $g(x) = f(x) + x$ satisfies the differential equation

$$g'(x) = 1 + \frac{1-(g(x)/x-1)^2}{1+x^2(g(x)/x-1)^2}.$$

This implies that $g'(x) > 0$ for all $x > 1$, so the limit $L_1 = \lim_{x\to+\infty} g(x)$ exists. In addition, we cannot have $L_1 < +\infty$, or else we would have $\lim_{x\to+\infty} g'(x) = 0$ whereas the differential equation forces this limit to be 1. Hence $g(x) \to +\infty$ as $x \to +\infty$.

Similarly, the function $h(x) = -f(x) + x$ satisfies the differential equation

$$h'(x) = 1 - \frac{1-(h(x)/x-1)^2}{1+x^2(h(x)/x-1)^2}.$$

This implies that $h'(x) \geq 0$ for all x, so the limit $L_2 = \lim_{x\to+\infty} h(x)$ exists. In addition, we cannot have $L_2 < +\infty$, or else we would have $\lim_{x\to+\infty} h'(x) = 0$ whereas the differential equation forces this limit to be 1. Hence $h(x) \to +\infty$ as $x \to +\infty$.

For some $x_1 > 1$, we must have $g(x), h(x) > 0$ for all $x \geq x_1$. For $x \geq x_1$, we have $|f(x)| < x$ and hence $f'(x) > 0$, so the limit $L = \lim_{x\to+\infty} f(x)$ exists. Once again, we cannot have $L < +\infty$, or else we would have $\lim_{x\to+\infty} f'(x) = 0$ whereas the original differential equation (e.g., in the form given in the first solution) forces this limit to be $1/(1+L^2) > 0$. Hence $f(x) \to +\infty$ as $x \to \infty$, as desired.

Third solution. (by Noam Elkies) Consider the function $g(x) = f(x) + \frac{1}{3}f(x)^3$, for which

$$g'(x) = f'(x)(1+f(x)^2) = 1 - \frac{f(x)^2}{x^2}$$

for $x > 1$. Since evidently $g'(x) < 1$, $g(x) - x$ is bounded above for x large. As in the first solution, $f(x)$ is bounded below for x large, so $\frac{1}{3}f(x)^3 - x$ is bounded above by some $c > 0$. For $x \geq c$, we obtain $f(x) \leq (6x)^{1/3}$.

Since $f(x)/x \to 0$ as $x \to +\infty$, $g'(x) \to 1$ and so $g(x)/x \to 1$. Since $g(x)$ tends to $+\infty$, so does $f(x)$. (With a tiny bit of extra work, one shows that in fact $f(x)/(3x)^{1/3} \to 1$ as $x \to +\infty$.)

□

7. ([B3] 2018) Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides 2^n-1, and $n-2$ divides 2^n-2.

Proof. The values of n with this property are 2^{2^ℓ} for $\ell = 1, 2, 4, 8$. First, note that n divides 2^n if and only if n is itself a power of 2; we may thus write $n = 2^m$ and note that if $n < 10^{100}$, then

$$2^m = n < 10^{100} < (10^3)^{34} < (2^{10})^{34} = 2^{340}.$$

Moreover, the case $m = 0$ does not lead to a solution because for $n = 1$, $n - 1 = 0$ does not divide $2^n - 1 = 1$; we may thus assume $1 \leq m \leq 340$.

Next, note that modulo $n-1 = 2^m - 1$, the powers of 2 cycle with period m (the terms $2^0, \dots, 2^{m-1}$ remain the same upon reduction, and then the next term repeats the initial 1); consequently, $n-1$ divides $2^n - 1$ if and only if m divides n, which happens if and only if m is a power of 2. Write $m = 2^\ell$ and note that $2^\ell < 340 < 512$, so $\ell < 9$. The case $\ell = 0$ does not lead to a solution because for $n = 2$, $n - 2 = 0$ does not divide $2^n - 2 = 2$; we may thus assume $1 \leq \ell \leq 8$.

Finally, note that $n - 2 = 2^m - 2$ divides $2^n - 2$ if and only if $2^{m-1} - 1$ divides $2^{n-1} - 1$. By the same logic as the previous paragraph, this happens if and only if $m - 1$ divides $n - 1$, that is, if $2^\ell - 1$ divides $2^m - 1$. This in turn happens if and only if ℓ divides $m = 2^\ell$, which happens if and only if ℓ is a power of 2. The values allowed by the bound $\ell < 9$ are $\ell = 1, 2, 4, 8$; for these values, $m \leq 2^8 = 256$ and

$$n = 2^m \leq 2^{256} \leq (2^3)^{86} < 10^{86} < 10^{100},$$

so the solutions listed do satisfy the original inequality.

□

8. ([B4] 2018) Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

Proof. We first rule out the case $|a| > 1$. In this case, we prove that $|x_{n+1}| \geq |x_n|$ for all n, meaning that we cannot have $x_n = 0$. We proceed by induction; the claim is true for $n = 0, 1$ by hypothesis. To prove the claim for $n \geq 2$, write

$$\begin{aligned} |x_{n+1}| &= |2x_n x_{n-1} - x_{n-2}| \\ &\geq 2|x_n||x_{n-1}| - |x_{n-2}| \\ &\geq |x_n|(2|x_{n-1}| - 1) \geq |x_n|, \end{aligned}$$

where the last step follows from $|x_{n-1}| \geq |x_{n-2}| \geq \cdots \geq |x_0| = 1$.

We may thus assume hereafter that $|a| \leq 1$. We can then write $a = \cos b$ for some $b \in [0, \pi]$. Let $\{F_n\}$ be the Fibonacci sequence, defined as usual by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. We show by induction that

$$x_n = \cos(F_n b) \qquad (n \geq 0).$$

Indeed, this is true for $n = 0, 1, 2$; given that it is true for $n \leq m$, then

$$\begin{aligned}
2x_m x_{m-1} &= 2\cos(F_m b)\cos(F_{m-1} b) \\
&= \cos((F_m - F_{m-1})b) + \cos((F_m + F_{m-1})b) \\
&= \cos(F_{m-2} b) + \cos(F_{m+1} b)
\end{aligned}$$

and so $x_{m+1} = 2x_m x_{m-1} - x_{m-2} = \cos(F_{m+1} b)$. This completes the induction.

Since $x_n = \cos(F_n b)$, if $x_n = 0$ for some n then $F_n b = \frac{k}{2}\pi$ for some odd integer k. In particular, we can write $b = \frac{c}{d}(2\pi)$ where $c = k$ and $d = 4F_n$ are integers.

Let x_n denote the pair (F_n, F_{n+1}), where each entry in this pair is viewed as an element of $\mathbb{Z}/d\mathbb{Z}$. Since there are only finitely many possibilities for x_n, there must be some $n_2 > n_1$ such that $x_{n_1} = x_{n_2}$. Now x_n uniquely determines both x_{n+1} and x_{n-1}, and it follows that the sequence $\{x_n\}$ is periodic: for $\ell = n_2 - n_1$, $x_{n+\ell} = x_n$ for all $n \geq 0$. In particular, $F_{n+\ell} \equiv F_n \pmod{d}$ for all n. But then $\frac{F_{n+\ell}c}{d} - \frac{F_n c}{d}$ is an integer, and so

$$\begin{aligned}
x_{n+\ell} &= \cos\left(\frac{F_{n+\ell}c}{d}(2\pi)\right) \\
&= \cos\left(\frac{F_n c}{d}(2\pi)\right) = x_n
\end{aligned}$$

for all n. Thus the sequence $\{x_n\}$ is periodic, as desired.

Remark. Karl Mahlburg points out that one can motivate the previous solution by computing the terms

$$x_2 = 2a^2 - 1, x_3 = 4a^3 - 3a, x_4 = 16a^5 - 20a^3 + 5a$$

and recognizing these as the Chebyshev polynomials T_2, T_3, T_5. (Note that T_3 was used in the solution of problem A3.)

Remark. It is not necessary to handle the case $|a| > 1$ separately; the cosine function extends to a surjective analytic function on $\mathbb{C}$ and continues to satisfy the addition formula, so one can write $a = \cos b$ for some $b \in \mathbb{C}$ and then proceed as above.

□

9. ([B5] 2018) Let $f = (f_1, f_2)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{1}{4}\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\right)^2 > 0$$

everywhere. Prove that f is one-to-one.

Proof. Let (a_1, a_2) and (a'_1, a'_2) be distinct points in $\mathbb{R}^2$; we want to show that $f(a_1, a_2) \neq f(a'_1, a'_2)$. Write $(v_1, v_2) = (a'_1, a'_2) - (a_1, a_2)$, and let $\gamma(t) = (a_1, a_2) + t(v_1, v_2)$, $t \in [0, 1]$, be the path between (a_1, a_2) and (a'_1, a'_2). Define a real-valued function g by $g(t) = (v_1, v_2) \cdot f(\gamma(t))$. By the Chain Rule,

$$f'(\gamma(t)) = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Abbreviate $\partial f_i/\partial x_j$ by f_{ij}; then

$$\begin{aligned} g'(t) &= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= f_{11}v_1^2 + (f_{12} + f_{21})v_1 v_2 + f_{22}v_2^2 \\ &= f_{11}\left(v_1 + \frac{f_{12} + f_{21}}{2f_{11}}v_2\right)^2 + \frac{4f_{11}f_{22} - (f_{12} + f_{21})^2}{4f_{11}}v_2^2 \\ &\geq 0 \end{aligned}$$

since f_{11} and $f_{11}f_{22} - (f_{12} + f_{21})^2/4$ are positive by assumption. Since the only way that equality could hold is if v_1 and v_2 are both 0, we in fact have $g'(t) > 0$ for all t. But if $f(a_1, a_2) = f(a'_1, a'_2)$, then $g(0) = g(1)$, a contradiction.

Remark. A similar argument shows more generally that $f : \mathbb{R}^n \to \mathbb{R}^n$ is injective if at all points in $\mathbb{R}^n$, the Jacobian matrix Df satisfies the following property: the quadratic form associated to the bilinear form with matrix Df (or the symmetrized bilinear form with matrix $(Df + (Df)^T)/2$) is positive definite. In the setting of the problem, the symmetrized matrix is

$$\begin{pmatrix} f_{11} & (f_{12} + f_{21})/2 \\ (f_{12} + f_{21})/2 & f_{22} \end{pmatrix},$$

and this is positive definite if and only if f_{11} and the determinant of the matrix are both positive (Sylvester's criterion). Note that the assumptions that $f_{12}, f_{21} > 0$ are unnecessary for the argument; it is also easy to see that the hypotheses $f_{11}, f_{12} > 0$ are also superfluous. (The assumption $f_{11}f_{22} - (f_{12} + f_{21})^2 > 0$ implies $f_{11}f_{22} > 0$, so both are nonzero and of the same sign; by continuity, this common sign must be constant over all of $\mathbb{R}^2$. If it is negative, then apply the same logic to $(-f_1, -f_2)$.)

$\square$