Fall 2021

Wednesday, November 24nd, 2021

Last time we started working on the following 5 problems. For some of them we found solutions, but for some we did not. Let's pick up from where we left ! I am adding more problems to this working sheet, hopping they will be a good exercise for the coming up exam!

1. Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

Proof. Substitute a/x for x in the given equation:

$$f'(x) = \frac{a}{xf(a/x)}.$$

Differentiate:

$$f''(x) = -\frac{a}{x^2 f(a/x)} + \frac{a^2 f'(a/x)}{x^3 f(a/x)^2}.$$

Now substitute to eliminate evaluations at a/x:

$$f''(x) = -\frac{f'(x)}{x} + \frac{f'(x)^2}{f(x)}.$$

Clear denominators:

$$xf(x)f''(x) + f(x)f'(x) = xf'(x)^2.$$

Divide through by $f(x)^2$ and rearrange:

$$0 = \frac{f'(x)}{f(x)} + \frac{xf''(x)}{f(x)} - \frac{xf'(x)^2}{f(x)^2}.$$

The right side is the derivative of $xf'(x)/f(x)$, so that quantity is constant. That is, for some d,

$$\frac{f'(x)}{f(x)} = \frac{d}{x}.$$

Integrating yields $f(x) = cx^d$, as desired.

□

2. Suppose that f is a function on the interval $[1, 3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,dx$ be?

Solution: In all solutions, we assume that the function f is integrable.

Proof. Let $g(x)$ be 1 for $1 \le x \le 2$ and -1 for $2 < x \le 3$, and define $h(x) = g(x) - f(x)$. Then $\int_1^3 h(x)\,dx = 0$ and $h(x) \ge 0$ for $1 \le x \le 2$, $h(x) \le 0$ for $2 < x \le 3$. Now

$$\begin{aligned}\int_1^3 \frac{h(x)}{x}\,dx &= \int_1^2 \frac{|h(x)|}{x}\,dx - \int_2^3 \frac{|h(x)|}{x}\,dx \\ &\ge \int_1^2 \frac{|h(x)|}{2}\,dx - \int_2^3 \frac{|h(x)|}{2}\,dx = 0,\end{aligned}$$

and thus $\int_1^3 \frac{f(x)}{x}\,dx \le \int_1^3 \frac{g(x)}{x}\,dx = 2\log 2 - \log 3 = \log\frac{4}{3}$. Since $g(x)$ achieves the upper bound, the answer is $\log\frac{4}{3}$.

□

3. Let $f : [0,1] \to \mathbb{R}$ be a continuous function with the property that

$$\int_0^1 f(x)\,dx = \frac{\pi}{4}.$$

Prove that there exists $x_0 \in (0,1)$ such that

$$\frac{1}{1+x_0} < f(x_0) < \frac{1}{2x_0}.$$

Proof. to come! □

4. Let $f : [0,\infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that

$$\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\,dx$$

diverges.

5. Let $f : [0,1] \to \mathbb{R}$ be a function for which there exists a constant $K > 0$ such that $|f(x) - f(y)| \le K\,|x-y|$ for all $x, y \in [0,1]$. Suppose also that for each rational number $r \in [0,1]$, there exist integers a and b such that $f(r) = a + br$. Prove that there exist finitely many intervals $I_1, \ldots, I_n$ such that f is a linear function on each I_i and $[0,1] = \bigcup_{i=1}^n I_i$.

6. Let $f : (1,\infty) \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - f(x)^2}{x^2(f(x)^2+1)} \qquad \text{for all } x > 1.$$

Prove that $\lim_{x\to\infty} f(x) = \infty$.

7. Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$.

8. Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

9. Let $f = (f_1, f_2)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{1}{4}\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\right)^2 > 0$$

everywhere. Prove that f is one-to-one.