Binomial coefficients and generating functions

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Binomial coefficients

The binomial coefficients \( \binom{n}{k} \) counts the number of ways one can choose \( k \) objects from given \( n \). They are coefficients in the binomial expansion

\[
(x + 1)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} + \cdots + \binom{n}{n-1} x + \binom{n}{n}.
\]

More explicitly,

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.
\]

Some simple but useful formulas for the binomial coefficients are

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

and

\[
\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}.
\]

Exercise: prove the above two formulas using the binomial expansion.

Generating functions

The terms of a sequence \((a_n)_{n \geq 0}\) can be combined into a function

\[
G(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots,
\]

called the generating function of the sequence. For example, the finite sequence \( \binom{m}{n} \), with \( m \) fixed and \( n \) varies, gives the function \((x+1)^m\). The generating function for \( a_n = \frac{1}{n} \) is \(-\ln(1-x)\).

Generating functions provide a method to understand recursive relations of a sequence.

**Theorem.** Suppose \( a_n (n \geq 0) \) is a sequence satisfying a second-order linear recurrence,

\[
a_n + u a_{n-1} + v a_{n-2} = 0.
\]

Suppose that the quadratic equation \( \lambda^2 + u \lambda + v = 0 \) has two distinct roots \( r_1, r_2 \). Then

\[
a_n = \alpha r_1^n + \beta r_2^n
\]

for some real numbers \( r_1, r_2 \).

**Proof.** Let \( G(x) = a_0 + a_1 x + a_2 x^2 + \cdots \) be the generating function of \( a_n \). The recurrence relation of \( a_n \) implies that

\[
G(x) - a_0 - a_1 x + u(G(x) - a_0) + v x^2 G(x) = 0.
\]
Solving for \( G(x) \), we have

\[
G(x) = \frac{a_0 + (ua_0 + a_1)x}{1 + ux + vx^2} = \frac{a_0 + (ua_0 + a_1)x}{(1 - r_1x)(1 - r_2x)}.
\]

Using partial fractions, we have

\[
G(x) = \frac{a_0 + (ua_0 + a_1)x}{(1 - r_1x)(1 - r_2x)} = \frac{\alpha}{1 - r_1x} + \frac{\beta}{1 - r_2x} = \sum_{n=1}^{\infty} (\alpha r_1^n + \beta r_2^n) x^n.
\]

Therefore, \( a_n = \alpha r_1^n + \beta r_2^n \).

1. Prove the identity

\[
\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}.
\]

2. Give two proofs of the identity

\[
\sum_{j=0}^{n} 2^{n-j} \binom{n}{j} \left( \left\lfloor \frac{j}{2} \right\rfloor \right) = \binom{2n+1}{n}.
\]

More Problems

1. (Putnam and beyond 880) Prove the identity

\[
\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}.
\]

2. (Putnam and beyond 867) Find the general formula for the sequence \((y_n)_{n \geq 0}\) with \( y_0 = 1 \) and \( y_n = ay_{n-1} + b^n \) for \( n \geq 1 \), where \( a \) and \( b \) are two fixed distinct real numbers.

3. (Putnam and beyond 872) Prove that the Fibonacci numbers \( F_n \) satisfy

\[
F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots.
\]

4. (Putnam and beyond 860) Consider the triangular \( n \times n \) matrix

\[
A = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 1 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
\]

Compute the matrix \( A^k \) for \( k \geq 1 \).

5. (Putnam 1992 B-2) Show that the coefficient of \( x^k \) in the expansion of \((1 + x + x^2 + x^3)^n\) is

\[
\sum_{j=0}^{k} \binom{n}{k} \binom{n}{k-2j}.
\]

6. Find the generating function of the sequence \( u_n = \text{number of nonnegative solutions of } 2a + 5b = n \).
7. (Putnam 2017 B-3) Suppose that \( f(x) = \sum_{i=0}^{\infty} c_i x^i \) is a power series for which each coefficient \( c_i \) is 0 or 1. Show that, if \( f(2/3) = 3/2 \), then \( f(1/2) \) must be irrational.

8. (Putnam 1962 A-5) Find

\[
\binom{n}{1}^2 + \binom{n}{2}^2 + \binom{n}{3}^2 + \cdots + \binom{n}{n}^2.
\]

9. (Putnam 2000 B-5) Let \( S_0 \) be a finite set of positive integers. We define finite sets \( S_1, S_2, \ldots \) of positive integers as follows. The integer \( a \) is in \( S_{n+1} \) if and only if exactly one of \( a - 1 \) or \( a \) is in \( S_n \). Show that there are infinitely many integers \( N \) for which

\[
S_N = S_0 \cup \{ N + a \mid a \in S_0 \}.
\]

10. (Putnam 2003 A-6) For a set \( S \) of nonnegative integers, let \( r_S(n) \) denote the number of ordered pairs \( (s_1, s_2) \) such that

\[
s_1, s_2 \in S, s_1 \neq s_2, \quad \text{and} \quad s_1 + s_2 = n.
\]

Is it possible to partition the nonnegative integers into two sets \( A \) and \( B \) in such a way that \( r_A(n) = r_B(n) \) for all \( n \)?

11. For positive integer \( n \), denote by \( S(n) \) the number of choices of the signs \( + \) or \( - \) such that \( \pm 1 \pm 2 \pm \cdots \pm n = 0 \). Prove that

\[
S(n) = \frac{2^{n-1}}{\pi} \int_0^{2\pi} \cos t \cos 2t \cdots \cos nt \, dt.
\]