Number theory problems

September 23, 2025

Problems

1. (Putnam 2021 A1) How many positive integers N satisfy all of the following three conditions?
   (a) N is divisible by 2020.
   (b) N has at most 2020 decimal digits.
   (c) The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.
2. (Putnam 1975 A1) A triangular number is a positive integer of the form $n(n+1)/2$. Show that m is a sum of two triangular numbers iff $4m+1$ is a sum of two squares.
3. Let $a_n = 10 + n^2$ for $n \geq 1$. For each n, let d_n denote the gcd of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.
4. (Putnam 1988 B1) If $n > 3$ is not prime, show that we can find positive integers a, b, c, such that $n = ab + bc + ca + 1$.
5. (Putnam 2024 A1) Determine all positive integers n for which there exist positive integers a, b, and c satisfying
$$2a^n + 3b^n = 4c^n.$$
6. Prove that for any integer $n > 1$, $n^4 + 4^n$ is not a prime number.
7. (Putnam 2019 A1) Determine all possible values of the expression
$$A^3 + B^3 + C^3 - 3ABC.$$
   where A, B, and C are nonnegative integers.
8. Prove that the equation
$$x^2 + y^2 + z^2 + 3(x + y + z) + 5 = 0$$
   has no solutions in rational numbers.
9. (Putnam 2007 B1) Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.
10. (Putnam 2003 B3) Show that for each positive integer n,
$$n! = \prod_{i=1}^{n} \mathrm{lcm}\left\{1, 2, \cdots, \lfloor \frac{n}{i} \rfloor\right\}.$$
   Here lcm denotes the least common multiple.

11. (Thanks to Tomasz Sobkowicz-Oliveira) Let a, b be positive integers such that $b > 1$ and $a^2 + ab$ is the product of two consecutive integers. Prove that

$$b \geq 1 + \sqrt{4a + 1}$$

and find all such pairs (a, b) for which $b = 1 + \sqrt{4a + 1}$.

12. (Putnam 2015 A2) Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

13. (Putnam 2013 A2) Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minumum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

14. (Putnam 2008 B4) Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \ldots, h(p^2 - 1)$ are distinct modulo p^2. Show that $h(0), h(1), \ldots, h(p^3 - 1)$ are distinct modulo p^3.