Geometry

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Vectors and complex numbers in geometry

1. (Putnam and beyond 607) A straight line cuts the asymptotes of a hyperbola in points A and B and the hyperbola itself in P and Q. Prove that $|AP| = |BQ|$.

2. (Putnam and beyond 577) The vectors $\vec{a}, \vec{b}$ and $\vec{c}$ in $\mathbb{R}^3$ satisfy

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a} \neq 0.$$

   Prove that $\vec{a} + \vec{b} + \vec{c} = 0$.

3. (Putnam and beyond 611) Find all regular polygons that can be inscribed in an ellipse with unequal semiaxes.

4. (Putnam 1967 B1) A hexagon is inscribed in a circle radius 1. Alternate sides have length 1. Show that the midpoints of the other three sides form an equilateral triangle.

5. (Putnam 1955 A2) Assume that O is the center of a regular n-gon $P_1 P_2 \ldots P_n$ and X is a point outside the n-gon on the line OP_1. Show that

$$|XP_1| \cdot |XP_2| \cdots |XP_n| + |OP_1|^n = |OX|^n.$$

Inequalities related

6. (Putnam and beyond, page 214) Let P be a point on the hyperbola $xy = 4$, and Q a point on the ellipse $x^2 + 4y^2 = 4$. Prove that the distance from P to Q is greater than 1.

7. (Putnam 1996 A2) Let C_1 and C_2 be circles whose centers are 10 units apart and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exist points X on C_1 and Y on C_2 such that M is the midpoint of the lines segment XY.

8. (Putnam 2012 A1) Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval (1, 12). Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

9. (Putnam 2016 B3) Supppose that S is a finite set of points in the plane such that the area of triangle $\triangle ABC$ is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.