Combinatorics

Botong Wang

March 2, 2022

Euler's formula for planar graphs

First, we recall that a planar graph is a graph embedded in the plane in such a way that edges do not cross. A planar graph divides the plane into regions, each of which is called a face.

Theorem. *Given a connected planar graph denote by V the number of vertices, by E the number of edges, and by F the number of faces (including the infinite face). Then*

$$V - E + F = 2.$$

The elementary proof of Euler's theorem uses induction on the number of edges. However, the theorem also has a deeper interpretation using topology, more specifically, Euler characteristics. The topological approach also applies to graphs embedded in a more general surface.

1. Is it possible to draw paths connecting each pair of 5 houses such that no two paths intersect?

2. Three conflicting neighbors have three common wells. Can one draw nine paths connecting each of the neighbors to each of the wells such that no two paths intersect?

The above two results are the key ingredients in the characterization of all planar graphs. In fact, the Kuratowski's theorem says the following:

> A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$.

Further exercises

1. (Putnam 1953 A2) The complete graph with 6 points and 15 edges has each edge colored red or blue. Show that we can find 3 points such that the 3 edges joining them are the same color.

2. (Putnam 1956 B5, Mantel's theorem) Show that a graph with $2n$ points and $n^2 + 1$ edges necessarily contains a 3-cycle, but that we can find a graph with $2n$ points and n^2 edges without a 3-cycle.

3. (Putnam 1968 A3) Let S be a finite set, and let P be the set of all subsets of S. Show that we can label the elements of P as A_i, such that $A_1 = \emptyset$ and for each $n \geq 1$, either $A_{n-1} \subset A_n$ and $|A_n - A_{n-1}| = 1$, or $A_{n-1} \supset A_n$ and $|A_{n-1} - A_n| = 1$.

4. (Putnam 1957 A5) Let S be a set of n points in the plane such that the greatest distance between two points of S is 1. Show that at most n pairs of points of S are a distance 1 apart.

5. At a homecoming dance, no boy dances with every girl, but each girl dances with at least one boy. Prove that there are two couples, gb and $g'b'$, who dance, such that g doesn't dance with b' and g' doesn't dance with b.

6. A finite group with n elements is generated by g and h. Can we arrange two copies of the elements of the group in a sequence (total length $2n$) so that each element is g or h times the previous element and the first element is g or h times the last?