Inequalities

November 3, 2021

Some theorems

Theorem (Inequality of arithmetic and geometric means). *For a sequence of nonnegative real numbers $x_1, x_2, \ldots, x_n$,*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdot \cdots \cdot x_n},$$

and the equality holds when all x_i are equal.

The simplest examples can also be written as

$$a^2 + b^2 \geq 2ab, \quad a^3 + b^3 + c^3 \geq 3abc.$$

A variant of the AM-GM inequality is the following form of Young's inequality.

Theorem (Young's inequality). *If a, b are nonnegative real numbers and if $p, q > 1$ are real numbers such that $1/p + 1/q = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

and equality holds when $a^p = b^q$.

A continuous form of the Young's inequality is the Hölder's inequality.

Theorem (Hölder's inequality). *Let $f(x), g(x)$ be continuous functions on $[a, b]$. If $p, q > 1$ are real numbers such that $1/p + 1/q = 1$, then*

$$\int_a^b |f(x)g(x)|dx \leq \Big(\int_a^b |f(x)|^p dx \Big)^{1/p} \Big(\int_a^b |g(x)|^q dx \Big)^{1/q}.$$

Remark. The Hölder's inequality holds on any measurable functions f, g on a measure space. For example, bounded open or closed subsets of $\mathbb{R}^n$.

Problems

1. Show that if $a_1, a_2, \ldots, a_n$ are nonnegative numbers, then

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq \big(1 + \sqrt[n]{a_1 a_2 \cdots a_n}\big)^n.$$

2. Let $a_1, a_2, \ldots, a_n$ be real numbers greater than 1. Prove the inequality

$$\sum_{i=1}^{n} \frac{1}{1 + a_i} \geq \frac{n}{1 + \sqrt[n]{a_1 a_2 \cdots a_n}}.$$

3. Prove that for any natural number $n \geq 2$ and any complex number x with $|x| \leq 1$,

$$|1 + x|^n + |1 - x|^n \leq 2^n.$$

4. Let f be a real-valued continuous function on $\mathbb{R}$ satisfying

$$f(x) \leq \frac{1}{2h} \int_{x-h}^{x+h} f(y)dy, \quad \text{for all } x \in \mathbb{R} \text{ and } h > 0.$$

Prove that (a) the maximum of f on any closed interval is assumed at one of the endpoints, and (b) the function f is convex.

5. Let $f : [0, \infty) \to [0, \infty)$ be a continuous, strictly increasing function with $f(0) = 0$. Prove that

$$\int_0^a f(x)dx + \int_0^b f^{-1}(b)dx \geq ab$$

for all positive numbers a and b, with equality if and only if $b = f(a)$. Here f^{-1} denotes the inverse function of f.

6. Find the maximal value of the ratio

$$\left(\int_0^3 f(x)dx\right)^3 \Big/ \int_0^3 f^3(x)dx,$$

as f ranges over all positive continuous functions on $[0, 1]$.

7. (2004 B2) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

8. (2003 A4) Suppose that a, b, c, A, B, C are real numbers with $a \neq 0$ and $A \neq 0$, such that

$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$$

for all real numbers x. Show that

$$|b^2 - 4ac| \leq |B^2 - 4AC|.$$

9. (2003 B6) Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)|dxdy \geq \int_0^1 |f(x)|dx.$$

10. (2004 A6) Suppose that $f(x, y)$ is a continuous real-valued function on the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y)dx\right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)dy\right)^2 dx \leq$$
$$\left(\int_0^1 \int_0^1 f(x,y)dxdy\right)^2 + \int_0^1 \int_0^1 [f(x,y)]^2 dxdy$$