Integrals

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In general, there are two types of integral problems: equalities and inequalities. In the two lectures, we will focus on equalities. Let us start with some standard techniques computing integrals.

**Explore the symmetry**

**Example** (Putnam 1987 B1). Evaluate

\[
\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x) + \sqrt{\ln(x+3)}}} \, dx.
\]

**Example** (Putnam and Beyond, Problem 453). Compute the integral

\[
\int_{-1}^1 \frac{\sqrt{x}}{\sqrt{1-x} + \sqrt{1+x}} \, dx.
\]

**Find the right substitution**

**Example** (Putnam and Beyond, Problem 455). Let \( a \) and \( b \) be positive real numbers. Compute

\[
\int_a^b \frac{e^x - e^{\frac{b}{x}}}{x} \, dx.
\]

**Trigonometric identities can be helpful**

**Example** (Putnam and Beyond, Problem 458). Compute the integral

\[
\int_0^{\frac{\pi}{4}} \ln(1 + \tan x) \, dx.
\]

*Hint:* \( \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \).

**Riemann sums**

**Example** (Putnam and Beyond, Page 154). Denote by \( G_n \) the geometric mean of the binomial coefficients

\[
\left( \begin{array}{c} n \\ 0 \end{array} \right), \left( \begin{array}{c} n \\ 1 \end{array} \right), \ldots, \left( \begin{array}{c} n \\ n \end{array} \right).
\]

Prove that

\[
\lim_{n \to \infty} \sqrt[n]{G_n} = \sqrt{e}.
\]
Solution (Suggested by –). We want to show that
\[
\lim_{n \to \infty} \frac{1}{n(n+1)} \sum_{i=1}^{n} \log \left( \frac{n}{i} \right) = \frac{1}{2}.
\]

Use the Stolz–Cesàro theorem, the left-hand-side is equal to
\[
\lim_{n \to \infty} \frac{1}{2n} \sum_{i=1}^{n} \left( \log \left( \frac{n+1}{i} \right) - \log \left( \frac{n}{i} \right) \right) = \lim_{n \to \infty} \frac{1}{2n} \sum_{i=1}^{n} \left( \log \frac{n+1}{n+1-i} \right)
\]
\[
= - \lim_{n \to \infty} \frac{1}{2n} \sum_{i=1}^{n} \left( \log \frac{n+1-i}{n+1} \right)
\]
\[
= - \int_{0}^{1} \frac{1}{2} \log(1-x) dx
\]
\[
= - \int_{0}^{1} \frac{1}{2} \log x dx
\]
\[
= - \frac{1}{2} \left( x \log x - x \right) \bigg|_{0}^{1}
\]
\[
= - \frac{1}{2}.
\]

Here, using l’Hopital’s rule, we have
\[
\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = 0,
\]
which implies the last equality.

Further exercises

1. (Putnam and Beyond, Page 150) Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function. Prove that
\[
\int_{0}^{\pi} x f(\sin x) dx = \pi \int_{0}^{\pi} f(\sin x) dx.
\]

2. (Putnam and Beyond, Problem 457) Let \( a \) be a positive real number. Compute the integral
\[
\int_{0}^{a} \frac{dx}{x + \sqrt{a^2 - x^2}}.
\]

3. (Putnam 1980, A3) Evaluate
\[
\int_{0}^{\frac{\pi}{2}} \frac{dx}{1 + (\tan x)\sqrt{2}}.
\]
Hint: this problem is indeed similar to the problem 2.

4. (Putnam 1982, A3) Evaluate
\[
\int_{0}^{\infty} \frac{\arctan(\pi x) - \arctan(x)}{x} dx.
\]

5. (Putnam 1989, A2) Evaluate
\[
\int_{0}^{a} \int_{0}^{b} e^{\max\{b^2x^2, a^2y^2\}} dy dx,
\]
where \( a \) and \( b \) are positive.
6. (Putnam and Beyond, Problem 468) Compute
\[
\lim_{n \to \infty} \left( \frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right).
\]

7. (Putnam and Beyond, Problem 447) Compute the indefinite integral
\[
\int \frac{x^2 + 1}{x^4 - x^2 + 1} \, dx.
\]

8. (Putnam 1992, A2) Define \( C(\alpha) \) to be the coefficient of \( x^{1992} \) in the power series about \( x = 0 \) of \( (1 + x)^\alpha \). Evaluate
\[
\int_0^1 \left( C(-y - 1) \sum_{k=1}^{1992} \frac{1}{y + k} \right) \, dy.
\]

9. (Putnam 2016, A3) Suppose that \( f \) is a function from \( \mathbb{R} \) to \( \mathbb{R} \) such that
\[
f(x) + f \left( 1 - \frac{1}{x} \right) = \arctan x
\]
for all real number \( x \neq 0 \). Find
\[
\int_0^1 f(x) \, dx.
\]