Limits of sequences

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First, we review Cauchy's definition of limit of a sequence.

Definition

- *A sequence (x_n) **converges** to a finite number L if for every $\epsilon > 0$, there exists a positive integer N such that for every $n > N$, $|x_n - L| < \epsilon$. This is denoted by $\lim_{n\to\infty} x_n = L$.*
- *A sequence (x_n) **tends to infinity** if for every every $M > 0$, there exists a positive integer N such that for every $n > N$, $x_n > M$. This is denoted by $\lim_{n\to\infty} x_n = +\infty$.*

We also review some baisc theorems.

Theorem (The squeezing principle)

- *If $a_n \le b_n \le c_n$ for all n, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} = L$.*
- *If $a_n \le b_n$ for all n, and if $\lim_{n\to\infty} a_n = +\infty$, then $\lim_{n\to\infty} b_n = +\infty$.*

Theorem (Weierstrass' theorem)
A monotonic bounded sequence of real numbers is convergent.

Theorem (Cauchy's criterion for convergence)
A sequence (x_n) of points in $\mathbb{R}^n$ (or, in general, in a complete metric space) is convergent if and only if any $\epsilon > 0$, there is a positive integer N such that whenever $m, n > N$, $|x_m - x_n| < \epsilon$.

The following theorem can be derived from Cauchy's criterion for convergence.

Theorem
Let X be a closed subset of $\mathbb{R}^n$ (or, in general, in a complete metric space) and $f : X \to X$ a function with the property that

$$||f(x) - f(y)|| \le c||x - y||$$

for any $x, y \in X$, where $0 < c < 1$ is a constant. Then f has a unique fixed point in X.

Theorem (Cesàro mean)
Suppose (x_n) is a converging sequence of real numbers with limit L. Then as n tens to infinity, the limit of the average of the first n terms of (x_n) is also equal to L. In other words,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{1\le i\le n} a_i = L.$$

The following Cesàro–Stolz Theorem is a generalization of the above theorem, and it is the sequence analog of the l'Hôpital's rule.

Theorem (The Cesàro–Stolz Theorem)
Let (x_n) and (y_n) be two sequences of real numbers with (y_n) strictly positive, increasing and unbounded. If

$$\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = L,$$

then the limit

$$\lim_{n\to\infty} \frac{x_n}{y_n}$$

exists and is equal to L.

Example 1
Let (a_n) be a decreasing sequence of positive numbers converging to 0. Prove that the seres

$$S = a_1 - a_2 + a_3 - a_4 + \cdots$$

is convergent.

Example 2
Prove that

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}.$$

Part of the problem is to give a precise interpretation of the right side of the equation.

Example 3
Prove the following identity of Ramanujan

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}} = 3.$$

Example 4
Let a_0, b_0, c_0 be real numbers. Define the sequences $(a_n), (b_n), (c_n)$ recursively by

$$a_{n+1} = \frac{a_n+b_n}{2}, \quad b_{n+1} = \frac{b_n+c_n}{2}, \quad c_{n+1} = \frac{c_n+a_n}{2}.$$

Prove that the sequences are convergent and find their limits.

Example 5
Let p be a positive real number. Compute

$$\lim_{n\to\infty} \frac{1^p+2^p+\cdots+n^p}{n^{p+1}}.$$

More exercises about limits of sequences.

1. Compute

$$\lim_{n\to\infty} \left|\sin\left(\pi\sqrt{n^2+n+1}\right)\right|.$$

2. Prove that

$$\lim_{n\to\infty} n^2 \int_0^{\frac{1}{n}} x^{x+1}dx = \frac{1}{2}.$$

3. Prove that for $n \geq 2$, the equation $x^n + x - 1 = 0$ has a unique root in the interval $[0, 1]$. If x_n denotes this root, prove that the sequence (x_n) is convergent and find its limit.

4. Prove that the sequence

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1),$$

is convergent.

5. Let $S = \{x_1, x_2, \ldots, x_n, \ldots\}$ be the set of all positive integers that do not contain the digit 9 in their decimal representation. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{x_n} < 80.$$

6. Let c and x_0 be fixed positive numbers. Define the sequence (x_n) recursively by

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{c}{x_{n-1}}\right).$$

Prove that the sequence converges and that its limit is $\sqrt{c}$.

7. For an arbitrary number $x_0 \in (0, \pi)$ define recursively the sequence (x_n) by $x_{n+1} = \sin x_n$. Compute

$$\lim_{n\to\infty} \sqrt{n} x_n.$$

8. Prove that the number

$$\sum_{n=1}^{\infty} \frac{1}{2^{n^2}}$$

is irrational.