Putnam Club. Fall 2020

Problem session for October 28. Polynomials.

1. a, b, and c are the three roots of the polynomial $x^3 - 3x^2 + 1$. Find $a^3 + b^3 + c^3$.

2. Find all polynomials satisfying the functional equation
$$xP(x-1) = (x-20)P(x).$$

3. Let $P(x)$ and $Q(x)$ be monic polynomials of degree 10. It is known, that the equation $P(x) = Q(x)$ has no real roots. Prove that the equation $P(x+1) = Q(x-1)$ has at least one real root.

4. Let $P(x)$ be a polynomial of odd degree with real coefficients. Show that the equation $P(P(x)) = 0$ has at least as many real roots as the equation $P(x) = 0$, counted without multiplicities.

5. Let $P(x)$ be a polynomial and a_1, a_2, b_1, b_2 be some real numbers. Assume that for any real x we have $P(a_1x + b_1) + P(a_2x + b_2) = P(x)$. Prove that $P(x)$ has at least one real root.

6. Let $P(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$, where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number, and if $r_1 + r_2 \neq r_3 + r_4$, then r_1r_2 is a rational number.

7. Let $P(x)$ be a polynomial with integer coefficients such that $|P(3)| = |P(7)| = 1$. Prove that $P(x)$ does not have any integer roots.

8. Let $P(x)$ be a polynomial of degree $n > 3$ whose zeros $x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n$ are real. Prove that
$$P'\Big(\frac{x_1 + x_2}{2}\Big) \cdot P'\Big(\frac{x_{n-1} + x_n}{2}\Big) \neq 0.$$

9. Let $p(x)$ be a real polynomial that is non-negative for all real x. Prove that for some k, there are real polynomials $f_1(x), \ldots, f_k(x)$ such that $p(x) = f_1(x)^2 + f_2(x)^2 + \ldots + f_k(x)^2$.

10. Suppose $P(x)$ is a polynomial of degree 3. A valid operation consists of replacing the current polynomial $Q(x)$ by either $Q(x) + Q'(x)$ or $Q(x) - Q'(x)$. Prove that a sequence of valid operations starting with $P(x)$ will never result in getting the same polynomial again.

11. Find all polynomials $P(x)$, $Q(x)$ with real coefficients such that
$$P(x)Q(x+1) - P(x+1)Q(x) = 1.$$

12. Find, with proof, for which n there exists a polynomial $P(x)$ of degree n with real coefficients and a polynomial $Q(x)$, such that $P(x^2 + x + 1) = P(x)Q(x)$.

13. Country Fateland selects its 6-person IMO team in the following way. There are 13 candidates who take team selection test and score $a_1, a_2, \ldots, a_{13}$ points. The scores are assumed to be distinct. Also there is a secret polynomial $P(x)$ with real coefficients that measures "creative potential" of contests. The six students with the highest values of $P(a_i)$ go to the IMO. Unfortunately, Mr. Unfair, the team leader, has pre-determined the team before the test. Find, with proof, the smallest n such that Mr. Unfair can always justify his selection regardless of test results using a polynomial of degree no more than n.

Some theory

- A polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is called a **monic** if $a_n = 1$;
- **Fundamental Theorem of Algebra:** a polynomial of degree n with complex coefficients has exactly n complex roots counting the multiplicity;
- If a polynomial $P(x)$ has real coefficients, then the complex roots of $P(x)$ must occurs in conjugate pairs;
- **Viete's relations**

  From the fundamental theorem of algebra, it follows that a polynomial with complex coefficients
  $$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0, \quad a_n \neq 0$$
  can be factored as
  $$P(x) = a_n(x - x_1)(x - x_2)\ldots(x - x_n).$$
  Equating the coefficients of powers of x in the two expressions, we obtain

  $x_1 + x_2 + \ldots + x_n = -a_{n-1}/a_n;$
  $x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n = a_{n-2}/a_n \qquad \ldots\ldots\ldots x_1x_2\ldots x_n = (-1)^n a_0/a_n.$
- **Intermediate Value Theorem:** for any continuous function $f : \mathbb{R} \to \mathbb{R}$, if $f(a_1) = b_1$ and $f(a_2) = b_2$, then for any b between b_1 and b_2 we can find $a \in [a_1, a_2]$ such that $f(a) = b$;
- If a zero of $P(x)$ has multiplicity greater than 1, then it is also a zero of $P'(x)$, and the converse is also true;
- If $x_1, x_2, \ldots, x_n$ are the zeros of $P(x)$, then $\dfrac{P'(x)}{P(x)} = \dfrac{1}{x - x_1} + \dfrac{1}{x - x_2} + \ldots + \dfrac{1}{x - x_n}$.
- if $P(x)$ is a polynomial with real coefficients, then there is a root of $P'(x)$ between any two roots of $P(x)$ (in particular, if all roots of $P(x)$ are real, then so are those of $P'(x)$);
- **Lagrange interpolation formula:** there is a unique polynomial of degree n having any given values in any given $n+1$ points. If $P(x_i) = a_i$ for $i = 1, 2\ldots, n+1$, then
  $$P(x) = \sum_{i=1}^{n+1} a_i L_i(x),$$
  where
  $$L_i(x) = \frac{(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_{n+1})}{(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_{n+1})}.$$