HILBERT SCHEMES AND QUOT SCHEMES

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ABSTRACT. Very sketchy notes for a talk on Hilbert Schemes and Quot schemes for the algebraic stacks reading seminar.

1. The Quot and Hilbert Scheme

Let $f : X \to S$ be a finitely presented separated morphism of schemes, L on X an f-relatively ample line bundle, and $P \in \mathbb{Q}[z]$ a polynomial, and F a quasicoherent locally finitely presented sheaf on X.

We define a functor

$$\mathrm{Quot}^P(F/X/S) : (\mathrm{Sch}/S)^{op} \to \mathrm{Set} \text{ sending } S' \to S \text{ to } \left\{ \begin{pmatrix} \text{isomorphism classes of quotients of quasicoherent sheaves} \\ F_{S'} \twoheadrightarrow G \text{ with } G \text{ locally finitely presented} \\ \text{quasi-cohernet sheaf with proper} \\ \text{support over } S' \text{ and for all } s' \in S', \\ \text{the Hilbert polynomial of } G_{X_{s'}} \text{ is } P. \end{pmatrix} \right\}$$

Theorem 1.1. The functor $\mathrm{Quot}^P(F/X/S)$ is represented by a scheme which is quasiprojective over S. It is projective if X/S is proper.

Definition 1.2. When $F = O_X$, we denote it by $\mathrm{Hilb}^P_{X/S}$ and it is called the **Hilbert scheme**.

Example 1.3. We write out some examples and leave it as an exercise to check that these descriptions are correct.

(1) Suppose $X = S = \mathrm{Spec}(\mathbb{C})$ and $F = \widetilde{\mathbb{C}^n}$. If $P(z) := k$, then $\mathrm{Quot}^P(F/X/S) = \mathrm{Hilb}^P_{X/S}$ is the Grassmannian paramterizing k-dimensional subvector spaces of $\mathbb{C}^n$. As one might guess from this, the proof of Theorem 1.1. **utilizes a generalization of the Grassmannian.**

(2) The moduli of degree d curves in $\mathbb{P}^2$ appears as a Hilbert scheme. In particular, $X = \mathbb{P}^2_{\mathbb{C}}$, $S = \mathrm{Spec}(\mathbb{C})$, and one takes $P(z) = \binom{z+2}{2} - \binom{z-d}{2} = \frac{1}{2}(z+2)(z+1) - \frac{1}{2}(z-d)(z-d-1) = (d+2)z + 1 - \frac{1}{2}d - \frac{1}{2}d^2$. Then $\mathrm{Hilb}^P_{\mathbb{P}^2_{\mathbb{C}}/\mathbb{C}}$ is the moduli of degree d curves in $\mathbb{P}^2_{\mathbb{C}}$. One checks via the universal property that this is a projective space of sufficiently large dimension.

(3) One can also consider instead the Hilbert scheme of points on some varieties. These alone are already quite interesting. It is already a nice exercise to think about the Hilbert scheme of points for $\mathbb{A}^2_{\mathbb{C}}$.

Here are the facts about them due to Forgarty, Briançon, and Göttsche.

Theorem 1.4. If S is smooth a quasiprojective surface over $\mathbb{C}$, then $\mathrm{Hilb}_n S$, the Hilbert scheme of n points is smooth irreducible of dimension $2n$. The topological Euler characteristic has generating function

$$\sum_{n \geq 0} \chi_{top}(\mathrm{Hilb}_n S) q^n = \prod_{m \geq 0} \frac{1}{(1-q^m)^{\chi_{top}(X)}}.$$

One of the main applications of the above theorem is showing that for integral locally planar curves, the compactified Jacobian is irreducible with dimension $g(C)$. One can find more in Dori Bejleri's lecture notes.

(4) Hartshorne's book on Deformation Theory is all about the Hilbert scheme. It might be worthwhile to peruse the book if one wants some more information.

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For an expository account of the quot scheme and its construction, I find Nitsin Nitsure's notes here helpful. The short summary is that (1) one makes a reduction to the case of projective space (you'd need some assumptions on $X \to S$ for this), (2) use Castelnuovo-Mumford regularity to produce an embedding of the Quot scheme into the Grassmannian, and then (3) show that this embedding is relatively representable (this uses the existence of a flattening stratification) and since the Grassmannian is representable, the Quot-scheme must be as well.

Since I do not assume the audience has attended a previous seminar where we discussed these methods, let me just define the Grassmannian functor associated to a quasicoherent sheaf $\mathcal{E}$ on a scheme X:

$$\mathrm{Grass}(k > 0, \mathcal{E})(T) := \{\text{quotients of } \mathcal{E}_T \text{ which are locally free of rank } k\}$$

This functor is representable by a quasiprojective scheme, but if $\mathcal{E}$ is coherent, then it is represented by a projective scheme. One can recover the usual Grassmannian by taking X to be the spectrum of a field and $\mathcal{E}$ a vector spcae of dimension n.

2. Application to $\mathcal{M}_g$ for $g \geq 2$

Fix $g \geq 2$. Then $\mathcal{M}_g$ is defined to be the category fibred in groupoids and for each $S \in \mathrm{Sch}_{\acute{e}t}$, one has the groupoid of

$$\mathcal{M}_g(S) := \{f : C \to S \mid f \text{ smooth proper with connected genus } g \text{ curves for geometric fibres}\}.$$

Theorem 2.1. The fibred category $\mathcal{M}_g$ is a Deligne-Mumford stack for $g \geq 2$.

The proof is quite long and will not fit into a talk, so I will do some trimming of the details. Here's a quick summary of some of the steps.

1. First off, we already know $\mathcal{M}_g$ is a stack from Jeremy's talk.
2. The idea of the proof is to show that $\mathcal{M}_g$ is some quotient stack. The useful part here is that $g \geq 2$ is that the canonical bundle $\omega_{C/\mathbb{C}}$ of any such curve is ample and its third tensor powr is very ample. This works over any base (not just $\mathrm{Spec}(\mathbb{C})$) provided one makes the appropriate adjustments. Using this, one embed the curve into some projective space (dimension $5g-6$ is enough). After some more work, one essentially gets some closed subscheme $\widetilde{M_g}$ of some scheme arising from the Hilbert scheme. Meanwhile, there is still an action of $G = \mathrm{GL}_{5g-5}$ one must quotient out corresponding to the choice of embedding. This realizes $\mathcal{M}_g \cong [\widetilde{M_g}/G]$ as a quotient stack.
3. The last bit of work to do is to show that $\mathcal{M}_g$ is not just an algebraic stack, but a DM stack. If one refines the condition about algebraic stacks being DM stacks iff the diagonal is formally unramified to the situation of a quotient stack $[X/G]$ for X a scheme, then it turns out one only needs to show that the automorphism group schemes $\mathrm{Aut}_k(C)$ of a genus g curve over a field $k = k^{alg}$ is reduced.

Now here's a sketch of the proof which we package into a sequence of lemmas and I leave to the audience to read from Olsson's book.

Lemma 2.2. Let $(S, f : C \to S) \in \mathcal{M}_g(S)$. Let $L_{C/S}L = (\Omega^1_{C/S})^{\otimes 3}$. Then

1. The sheaf $f_*L_{C/S}$ is locally free of rank $5g-5$ on S.
2. The map $f^*f_* : L_{C/S} \to L_{C/S}$ is surjective and the resulting S-map
$$C \to \mathbb{P}(f_*L_{C/S})$$
is a closed embedding i.e. $L_{C/S}$ is f-very ample.
3. Formation of $f_*L_{C/S}$ commutes with base change i.e. if $g : S' \to S$ is a morphism, then the natural map
$$g^*f_*L_{C/S} \to f'_*L_{C'/S'}$$
is an isomorphism where $f' : C' := C \times_S S' \to S'$ is the base change of C/S.

Lemma 2.3. The functor

$$\widetilde{M_g} : \mathrm{Sch} \to \mathrm{Set}$$

which takes a scheme S to the set of isomorphism classes of pairs

$$(f : C \to S, \sigma : O_S^{5g-5} \cong f_*L_{C/S})$$

where $(S, f : C \to S) \in \mathcal{M}_g(S)$ is representable by a quasi-projective scheme. Here, an isomorphism of pairs is given by an isomorphism of curves s.t. the obvious diagram of sheaves commutes.

The proof of this lemma takes up the bulk of the work. In any case, there is an action on $\widetilde{M_g}$ by GL_{5g-5} which acts on an S-point $(C/S,\sigma)$ by $\gamma * (C/S,\sigma) \to (C/S,\sigma\circ\gamma)$ for $\gamma \in \mathrm{GL}_{5g-5}(S)$.

Lemma 2.4. There is an isomorphism $\mathcal{M}_g \cong [\widetilde{M_g}/G]$.

The proof of this lemma requires one to check that the map $\pi : \widetilde{M_g} \to \mathcal{M}_g$ given by $(C/S,\sigma) \mapsto (S,C)$ has

$$\widetilde{M_g} \times_{\mathcal{M}_g} S$$

as the G_S-torsor of isomorphisms $\sigma : O_S^{5g-5} \to f_* L_{C/S}$.

Lemma 2.5. If C/k is a smooth genus g curve over $k = \overline{k}$, then $\mathrm{Aut}_k(C)$ is a reduced group scheme. Consequently, the previous lemma implies $\mathcal{M}_g$ is a Deligne-Mumford stack.

Proof. It suffices to show that if $A' \to A$ is a surjective morphism of k-algebras with squarezero kernel, then the map

$$\mathrm{Aut}_k(C)(A') \to \mathrm{Aut}_k(C)(A)$$

is injective. Let I denote the kernel of $A' \to A$.

Suppose $\alpha \in \mathrm{Aut}_k(C)(A')$ and $\overline{\alpha} : C_A \to C_A$ is a fixed automorphism. Then we need to show α is uniquely determined. Look at the diagram

$$\begin{tikzcd}
C_A \arrow[r, "\overline{\alpha}"] \arrow[d, hook] & C_A \arrow[r, hook] & C_{A'} \arrow[d] \\
C_{A'} \arrow[rr] \arrow[urr, dotted] & & \mathrm{Spec}(k)
\end{tikzcd}$$

The space of such dotted arrows, which α lands in, forms a torsor under

$$\mathrm{Hom}(\alpha^*\Omega^1_{C_A/A}, I \otimes_A O_{C_A}) \cong H^0(C_A, \overline{\alpha}^* T_{C/A} \otimes I \otimes_A O_C)$$

The latter is zero since the tangent bundle has negative degree and so it is zero on every fibre. So the set of possible lifts α is singleton and the map is injective. $\square$

Remark 2.6. The situations where $g = 0, 1$ can also be considered, but they have a different flavor due to the behavior of $\Omega_{C/S}$ in these situations.

For $g = 0$, one has that $\mathcal{M}_0 \cong B_S\,\mathrm{PGL}_2$ so it isn't a DM stack but it is algebraic.

For $g = 1$, we saw before that $\mathcal{M}_1$ is not even a stack. But if one considers instead proper smooth *algebraic spaces* C/S with geometric fibres that are connected curves of genus 1, then one gets an algebraic stack. One can also consider the stack $\mathcal{M}_{1,1}$ classifying elliptic curves instead.

Remark 2.7. We won't have time to get to it this semester, but $\mathcal{M}_g$ has a compactification $\overline{\mathcal{M}_g}$ which is the moduli stack of stable curves. This is the result of Deligne-Mumford's famous paper on the subject.

3. $\mathrm{Bun}_n(X)$

In this section, we sketch the proof that the stack of rank n vector bundles is an algebraic stack of finite type if we are considering $f : X \to S$ s.t. f is projective and S is noetherian. We follow Laumon-Morét-Bailly's proof (theorem 4.6.2.1 in their book). Another approach is taken in Lieblich's paper and for $\mathrm{Bun}_G(X)$ being a smooth algebraic stack in the case of curves, see Wang's paper.

Theorem 3.1. $\mathrm{Bun}_n(X/S)$ is an algebraic stack of finite type if X/S is projective with S noetherian.

Proof. We actually prove instead that $\mathrm{Coh}_{X/S}$ is an algebraic stack of finite type. (In Laumon-Moret-Bailly it is claimed this is enough since $\mathrm{Bun}_n(X/S)$ is an open substack of $\mathrm{Coh}_{X/S}$).

The hard part is really just showing that $\mathrm{Coh}_{X/S}$ has a smooth atlas. This requires projectivity of X/S. First off, take the Quot scheme $\mathrm{Quot}_{O_X^n/X/S} := \coprod_{P\in\mathbb{Q}[z]} \mathrm{Quot}^P_{O_X^n/X/S}$. Let $\mathrm{Quot}^\circ_{O_X^n/X/S}$ denote the open subscheme given by the condition that $\alpha : O_U^N \to \mathcal{F}$ is in $\mathrm{Quot}^\circ_{O_X^n/X/S}(U)$ if and only if $R^p(f_U)_*\mathcal{F} = 0$ for $p > 0$ and $(f_U)_*\alpha : O_U^n \to (f_U)_*\mathcal{M}$ is a surjective morphism.

Now fix $O_X(1)$ a relatively ample line bundle for X/S. We have a map

$$P_{N,n} : \mathrm{Quot}^\circ_{O_X^n/X/S} \to \mathrm{Coh}_{X/S}$$

which sends any surjecitve morphism $O_U^N \to \mathcal{F}$ to $\mathcal{F}(-n)$. Then we get a map

$$\prod_{N\geq 0, n\geq 0} \mathrm{Quot}^{\circ}_{O_X^n/X/S} \to \mathrm{Coh}_{X/S}.$$

It suffices to show that this is surjective and smooth.

Let $U \to \mathrm{Coh}_{X/S}$ be a morphism associated to $\mathcal{M} \in \mathrm{Coh}_{X/S}(U)$ where we can assume U is an affine scheme over S. For fixed $N, n \geq 0$, one can find a largest open subscheme $U_{N,n} \subseteq U$ such that $R^p(f_U)_*\mathcal{F}(n) = 0$ for $p > 0$ and $f_U^*(f_U)_*\mathcal{F}(n) \to \mathcal{F}(n)$ is surjective and $f_{U*}\mathcal{M}(n)$ is locally free. In that case, the projection from the fibre product

$$U \times_{\mathrm{Coh}_{X/S}} \mathrm{Quot}^{\circ}_{O_X^n/X/S} \to U$$

factors through $U_{N,n}$ and its source is the GL_N-torsor

$$Isom(O^N_{U_{N,n}}, f_{U_{N,n}*}\mathcal{M}(n)).$$

So the projection map is surjective and smooth and we win because we have $U = \coprod_N U_{N,n}$ for $n \gg 0$.

Now let's show the "easier" parts i.e. checking that the diagonal is representable. Recall that representability of the diagonal is equivalent to showing that the Isom-sheaves are algebraic spaces. But in this situation, the Isom-sheaves are are schemes (do I need to provide more details). One only needs to show that if $u_1, u_2 \in \mathrm{Coh}_{X/S}(U)$ given by $\mathcal{M}, \mathcal{N}$, then the functor $Isom(u_1, u_2)$ sending $\varphi : V \to U$ to $Isom(\varphi^*\mathcal{M}, \varphi^*\mathcal{N})$ is representable by some scheme. The obvious choice is to pick the subscheme of $Hom(\mathcal{M}, \mathcal{N}) \times Hom(\mathcal{N}, \mathcal{M})$ corresponding to isomorphisms and consider the projection onto the first factor. □

Remark 3.2. It is necessary here to work with $f : X \to S$ projective. Otherwise, the proof of existence of a smooth surjective atlas as above does not work. I don't have an example offhand where (a) X is quasiprojective and (b) $\mathrm{Bun}_r(X)$ has not smooth surjective atlas.

Remark 3.3. As an advertisement for a summer topic, we can certainly try and learn more about Bun_G. Here are some things one might hope to cover

(1) Prove that $\mathrm{Bun}_G(X)$ is a smooth algebraic stack for X a smooth projective curve, and G a reductive group.
(2) Define the cotangent stack and study the Hitchin fibration. Basically, the tangent space to a point of $\mathrm{Bun}_G(X)$ is the data of a Higgs bundle.
(3) Discuss some other properties of $\mathrm{Bun}_G(X)$ such as its cohomology, line bundeles on $\mathrm{Bun}_G(X)$, the paper of Beauville-Laszlo on Conformal Blocks, etc.....(give the audience some references).