Representations of Semisimple Lie Algebras II

$\mathfrak{g}$ — semisimple Lie algebra over $k=\bar{k}$, char 0.

Today: 1) structure theory & classification of semisimple Lie alge.

2) rep'n theory

<u>Defn</u> We say a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is <u>Cartan subalgebra</u> if

1) it is nilpotent

2) self normalizing i.e. $\{X \in \mathfrak{g} \mid [X,Z] \in \mathfrak{h} \ \forall Z \in \mathfrak{h}\} = \mathfrak{h}$.

(already rules ~~out~~ out 0).

<u>Thm</u> Cartan subalgebras exist.

<u>Proof Idea</u>: Consider <u>centralizers</u> of various elements of $\mathfrak{g} \ni X$.

$\longrightarrow$ i.e. $\{Y \in \mathfrak{g} \mid [Y,X] = 0\} = \ker(ad_X)$

Consider ~~~~ dimensions of $\ker(ad_X)$ for all possible X.

Take a minimal one. This is the Cartan subalgebra.

<u>Proposition</u>: CSA's of semisimple algebras $\mathfrak{g}$ is abelian. Furthermore, a Cartan subalgebra is a maximal subalgebra consisting of semisimple elements. CSA's can be called "maximal toral".

<u>Thm</u>: $\mathfrak{g}$ semisimple $\Rightarrow$ any 2 CSA's are conjugate. So, $\exists\, x_1, x_2, x_3, \ldots, x_n$ in $\mathfrak{g}$ s.t. $e^{ad x_1} \cdots e^{ad x_n} \in GL(\mathfrak{g})$ and $\mathfrak{h}_1$ ~~&~~ & $\mathfrak{h}_2$ satisfy $e^{ad x_1} \cdots e^{ad x_n} \mathfrak{h}_1 = \mathfrak{h}_2$

$\mathfrak{h}$ = abelian & toral implies $\mathfrak{h}$ acts diagonally on $\mathfrak{g}$ through ad.

The root space decomposition wrt Cartan $\mathfrak{h}$ is

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \oplus \mathfrak{h} \quad \text{(but sum not over all } \mathfrak{h}^*)$$

The linear functionals appearing in the $\oplus$'s are the roots of $\mathfrak{g}$ & $\mathfrak{g}_\alpha = \{X \mid [H,X] = \alpha(H)X\}$

Facts:
- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ (actually = in semisimple case).
- $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, then $B(X,Y) = 0$ unless $\beta = -\alpha$.
- $B|_{\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}}$ is nondegenerate.

Let $\mathfrak{h}_\alpha := [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$. Pick $X_\alpha \in \mathfrak{g}_\alpha$, $Y_{-\alpha} \in \mathfrak{g}_{-\alpha}$ s.t. $B(X_\alpha, Y_{-\alpha}) \neq 0$. Consider $B([X_\alpha, Y_\alpha], H) = -\alpha(H) B(X_\alpha, Y_{-\alpha})$. (not obvious!). Then, $[X_\alpha, Y_{-\alpha}] \neq 0$ implies $\mathfrak{h}_\alpha \neq 0$.

$\exists H_\alpha \in \mathfrak{h}_\alpha$ nonzero. We may normalize so that $\alpha(H_\alpha) = 2$. Then the subalgebra spanned by $H_\alpha, X_\alpha, Y_\alpha$, $k\langle H_\alpha, X_\alpha, Y_\alpha\rangle \simeq \mathfrak{sl}_2$.

$\mathfrak{g}$ can be acted on by ad_{X_α}, $ad_{Y_{-\alpha}}$, ad_{H_α}. So, this is a rep'n of $\mathfrak{sl}_2$. Therefore $\mathfrak{g}$ splits as a direct sum of irred rep'n of $\mathfrak{sl}_2$, and $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{h}_\alpha \cong V(2)$. ($\Longrightarrow$ $\alpha \in \Phi$ implies $2\alpha \notin \Phi$).

So, the $\mathfrak{g}_\alpha$'s, $\mathfrak{h}_\alpha$'s are 1-dimensional. B/c wts of rep'n's of $\mathfrak{sl}_2$ are in $\mathbb{Z}$, all $\alpha(H_\beta) \in \mathbb{Z}$ (integer eigenvalues of $\mathfrak{sl}_2$-mods).

The set of α's is the root space denoted Φ.

$B|_{\mathfrak{h}}$ is nondegenerate. Can identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via B. Also, $\sum_{\mathfrak{h}_\alpha, \alpha \in \Phi} k\mathfrak{h}_\alpha = \mathfrak{h}$. Therefore, $\exists$ a subset Δ so that

$$\mathfrak{h} = \bigoplus_{\alpha \in \Delta} k \cdot H_\alpha.$$

Thm. One can choose Δ so that any $\alpha \in \Phi$ is a $\mathbb{Z}$-linear comb. of those in Δ. Call Δ the set of simple roots

$\Phi^+ :=$ positive $\mathbb{Z}$-comb. of $\Delta \cap \Phi$, H_α is called a coroot dual to α.
From here on, we fix Δ the simple roots.

Defn: a weight of $\mathfrak{g}$ is an element $\lambda \in \mathfrak{h}^*$.
a dominant weight is a λ s.t. $B(\lambda, \alpha) \geq 0 \quad \forall \alpha \in \Delta$
an integral weight is " " " $B(\lambda, \alpha) \in \mathbb{Z}$ " " "

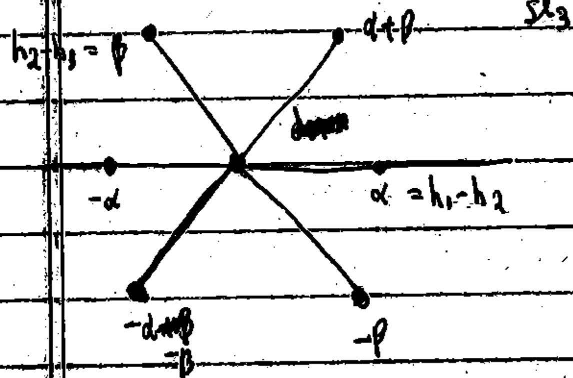
Think about exchanging α and $-\alpha$. If $\alpha \in \Phi$, then $\ker(\alpha)$ is the root hyperplane associated to α. The $\alpha \mapsto -\alpha$ can be thought of as reflection over the root hyperplane

$$S_\alpha(-) : \lambda \longmapsto \lambda - \frac{2B(\alpha,\lambda)}{B(\alpha,\alpha)}\lambda.$$

Defn: $W :=$ group generated by $\{s_\alpha\}_{\alpha \in \Delta}$ — Weyl group.
It's the isometry group of the space of roots Φ.

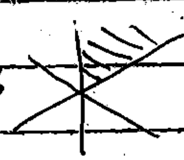
Defn: The dominant Weyl chamber is the set of dominant weights. (where Weyl vector sits).

Picture:



α, β simple, $\alpha+\beta$ is not.

dominant chamber is



$$\mathfrak{h} = \mathbb{C}\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix} \oplus \mathbb{C}\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}$$

$E_{ij} :=$ 1 in (i,j) zero otherwise.
Then $\mathfrak{h}$ acts on E_{ij} via $h_i - h_j$ where h_i = ith entry in diagonal.

$$\Phi = \{ h_i - h_j \mid i \neq j,\ i,j \in \{1,2,3\} \}$$

simple h_1-h_2, h_2-h_3

Thm: (Classification of Simple Lie Algebras)

Inner product of simple roots $\longleftrightarrow$ Dynkin diagrams.

$\rightarrow$ A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2

sl_{n+1}, $so(2n+1)$, sp_{2n}, $so(2n)$

Serre presentation goes from Dynkin Diagrams & goes to Lie alg.

Thm (Levi-Malcev): $\forall$ $\mathfrak{g}$ Lie alg, $\exists$ decomposition $\mathfrak{g} \cong \mathfrak{l} \oplus \mathfrak{r}$ where $\mathfrak{l}$ = semisimple and $\mathfrak{r}$ = solvable.

Defn The universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is an associative algebra $U\mathfrak{g}$ + linear map $\mathfrak{g} \to U\mathfrak{g}$ which is univ for

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr, "\text{ass. alg map}"'] & U\mathfrak{g} \arrow[d, "\exists!"] \\ & A \end{tikzcd}$$

So, $\mathfrak{g}$-repns $\longleftrightarrow$ $U(\mathfrak{g})$-reps.

Where $U(\mathfrak{g}) = T(\mathfrak{g}) / \langle x \otimes y - y \otimes x - [x,y] \rangle$.

Thm (PBW) $S(\mathfrak{g}) \xrightarrow{\sim} gr(U(\mathfrak{g}))$ via obv grading on $T(\mathfrak{g})$.

So if fix an ordering on monomials, then they span $U(\mathfrak{g})$ as a k-v.s.

If $\mathfrak{g}$ is semisimple, then $U(\mathfrak{g}) \simeq U(\mathfrak{n}^-) U(\mathfrak{h}) U(\mathfrak{n}^+)$.

and here,

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \quad , \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$$

Construction: Fix weight $\lambda \in \mathfrak{h}^*$. Consider $U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$ where $\mathfrak{n}^+$ acts (as raising op.) trivially and $\mathfrak{h}$ acts by λ $=: k_\lambda$ (one dim).

Make the following constr. $M_\lambda = U(\mathfrak{g}) \otimes_{\underbrace{U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)}_{U(\mathfrak{b})}} k_\lambda$ (Verma mod).

Thm
- M_λ has a ! simple quotient, $L(\lambda)$
- If dominant int λ, then $L(\lambda)$ is f.d.im
- All f.d. irreps of $\mathfrak{g}$ arise like this