Algebraic Spaces — Jameson Cluger

Def: C site, a class of objects $S \subseteq C$ is stable if $\forall \{U_i \to U\}_i$ covering,

$$U \in S \text{ iff } U_i \in S \ \forall i$$

A property P is stable if $S := \{\text{objects of } C \text{ w/ property } P\}$ is stable.

Def: A <u>closed subcategory</u> $D \subseteq C$ is a subcat s.t.

a) D contains all iso

b) $\forall$ Cartesian diagrams in C

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

$f \in D \Rightarrow f' \in D.$

Def: A closed subcategory $D \subseteq C$ is <u>stable</u> if for all $f: X \to Y$ in C, $\{Y_i \to Y\}$ covering in C, $f \in D$ iff $\underbrace{f_i : Y_i \times_Y X \to Y_i \text{ is in } D}_{\text{local on target}}$.

A stable closed subcat. $D \subseteq C$ is local on domain if $\forall f: X \to Y$ in C, $\{X_i \to X\}$ covering

$$f \in D \text{ iff } f_i : X_i \to X \to Y \in D.$$

Prop: $(S\text{-sch})_{\acute{e}t}$

(i) proper, separated, surjective, and quasicompact are stable.

(ii) stable and local on domain; locally of finite type, locally of finite presentation, flat, étale, universally open, locally quasifinite, smooth

Def: $f: F \to G$ morphism of sheaves on $(Sch/S)_{\acute{e}t}$

(i) f <u>representable by schemes</u> if in

$$\begin{tikzcd} V \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\square"] & G \arrow[d] \\ U \arrow[r] & F \end{tikzcd} \quad \text{Cartesian}$$

V is a scheme when U is a scheme

(ii) Let P be a stable property of morphisms of schemes.
<u>f rep by schemes has P</u> if

$\forall$ S-sch $T \longrightarrow G$,

$$T \times_G F \longrightarrow T \quad \text{has } P.$$

Rmk: F, G schemes so all maps are rep by schemes (aka fibre products of sch are sch).

Lemma: F sheaf on $(Sch/S)_{et}$ $\quad \Delta: F \longrightarrow F \times F$ is rep by schemes iff $\forall$ T schemes, $T \to F$ is rep by schemes.

$$\left\{ \begin{tikzcd} T \times T' \arrow[d] & T \times_F T' \arrow[l] \arrow[d] & \cong & T' \times_F T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d] \\ F \times F & F \arrow[l] & & T' \arrow[r] & F \end{tikzcd} \right\}.$$

Def: An <u>algebraic space</u> over S, is a functor

$$X : (Sch/S)_{\acute{e}t} \longrightarrow Sets$$

such that

i) X is an étale sheaf

ii) $\Delta_X : X \to X \times_S X$ is rep by schemes

iii) $\exists$ an étale surjective S-map $U \to X$ from an S-scheme U.

Morphisms of algebraic spaces are natural transformations. The cat. has finite limits hence fibre products.

Def: P stable, the X has P if $\exists$ étale surj $U \to X$ where U has P.

Def: P local on domain, $f : X \to Y$, f has P if $\exists$ étale covers $U \to X$, $V \to Y$ s.t.

$$U \times_Y V \longrightarrow V \text{ has property } P.$$

One can regard an alg space as a scheme mod an equivalence relation that's étale.

An <u>étale relation</u> is a monomorphism $R \hookrightarrow X \times_S X$ s.t.

(i) $\forall$ S-sch T, $R(T) \hookrightarrow X(T) \times_{S(T)} X(T)$ is an equiv rel

(ii) $s,t : R \to X \times X \to X$ are étale

The associated algebraic space is to take the presheaf

$$(Sch/S)_{ét} \longrightarrow Sets$$

$$T \longmapsto X(T)/R(T)$$

and sheafify. This is denoted X/R and has étale cover X.

If $Y \to X$ étale surj and Y scheme, X alg space the

$$X \cong Y/R \text{ for } R := Y \times_X Y \hookrightarrow Y \times_S Y.$$

Expl: X scheme, G discrete group $\curvearrowright X$ freely.
i.e. $G \times X \to X \times X$ $(g, x) \mapsto (x, g \cdot x)$ is a mono

X/G = sheaf associated to $T \mapsto X(T)/G(T)$.

Expl: k char 0 field, $\mathbb{Z} \curvearrowright \mathbb{A}^1_k$ by $x \mapsto x+n$.

Then $\mathbb{A}^1_k/\mathbb{Z}$ is an algebraic space. <u>But</u> it's not a scheme.

$$\text{B/c} \quad \begin{array}{ccc} \mathbb{A}^1_k \twoheadrightarrow & X & \Rightarrow \mathcal{O}_X(U) \hookrightarrow k(x)^{\mathbb{Z}} \\ & \uparrow & \qquad\qquad \cong \\ & U & \qquad\qquad k \end{array}$$

force $U = pt$ ↯.

Expl: $S = \mathrm{Spec}(\mathbb{Q})$

$U = \mathrm{Spec}(\overline{\mathbb{Q}})$

$G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

$$X = \mathrm{Spec}(\overline{\mathbb{Q}})/G \longrightarrow S = \mathrm{Spec}\,\mathbb{Q}.$$

$\uparrow$

algebraic space.