Stacks Seminar – Scribe Notes for Jeremy N.'s Talk 3/21/25

Descent Theory – Faithfully Flat Descent and the Stack Condition

n.b. we will work with the affine site $(Aff/S)_{fppf}$
it gives the same sh.(see Olsson; Stacks §34.7), to $(Sch/S)_{fppf}$.

Motivation: Glueing Lemma for Sheaves on Top Space X.

Recall def of Sheaf on X: a contravariant functor $Open(X) \xrightarrow{F} Set$ satisfying two conditions

i) (Separation) For $\coprod_{i\in I} U_i \twoheadrightarrow U$ open cover, $s,t \in F(U)$ and s_i, t_i be image of s,t in $F(U_i)$.

If $s_i = t_i \ \forall i$, then $s=t$. (called sep presheaf)

ii) (gluing) For such coverings, $s_i \in F(U_i)$ we had

$s_{ij} \triangleq s_i|_{U_i\cap U_j}$, $\quad s_{ij} = s_{ji}$

then $\exists!$ $s\in F(U)$ s.t. $s|_{U_i} = s_i$.

Gluing Lemma: If $\coprod U_i$ covering of X, F_i sheaves on U_i, st

- $\varphi_{ij} : F_i|_{U_i\cap U_j} \xrightarrow{\sim} F_j|_{U_i\cap U_j}$
- and $\varphi_{ik} = \varphi_{jk}\circ\varphi_{ij}$ on $U_i\cap U_j\cap U_k$,

then $\exists!$ F on X sheaf s.t. $F|_{U_i} = F_i$.

this must change in F.C. setting.

Remark: We want to generalize to fibred categories on a site, in particular, to do this to QCoh. "gluing" is what we call "descent".

Descent for modules

Lemma: modules form a sheaf on $(\mathrm{Aff}/\mathrm{Spec}\,A)_{fppf}$.

$F: A \to B$ is fppf, M an A-module, then

$$(*) \qquad 0 \to M \xrightarrow{m \mapsto m \otimes f(1)} M \otimes_A B \xrightarrow{p_1 - p_2} M \otimes_A B \otimes_A B \text{ is exact.} \quad A\text{-modules}$$

(Reminder: faithfully flat means $0 \to M_1 \to M_2 \to M_3 \to 0$ exact A-mod
$\Leftrightarrow 0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A C \to 0$)

Here, $p_1(m \otimes b) = m \otimes b \otimes 1$
$p_2(m \otimes b) = m \otimes 1 \otimes b$

(Grothendieck's trick) ?

Proof: Suppose a section g of F exists i.e. $g \circ F = id_A$.

Then $(*)$ exact at M i.e. $M \hookrightarrow M \otimes_A B$.

Exactness at $M \otimes B$. Say $m \otimes b \in \ker(p_1 - p_2)$. Then

$$m \otimes b \otimes 1 - m \otimes 1 \otimes b = 0$$
$$\Leftrightarrow m \otimes (b \otimes 1 - 1 \otimes b) = 0$$
$$\Leftrightarrow b \otimes 1 - 1 \otimes b = 0$$
$$\Rightarrow b \in \mathrm{Im} \text{ of } A.$$

* exercise

$\underset{b/c}{\Rightarrow}$ $0 \to A \to B \to B \otimes_A B$ is exact.

∎

Corollary: (Descent for modules) $F: A \to B$ ff.

M' - B-module, $\varphi : M' \otimes_A B \xrightarrow{\sim} B \otimes_A M'$

$$B \otimes M' \otimes B \underset{\varphi_1}{\xrightarrow{\sim}} B \otimes B \otimes M' \underset{\varphi_2}{\xleftarrow{\sim}} M' \otimes B \otimes B \underset{\varphi_3}{\longrightarrow} B \otimes M' \otimes B$$

Suppose $\varphi_1 \circ \varphi_3 = \varphi_2$. Then $\exists!$ A-mod M' s.t. $M \otimes B = M'$.

Pictorially:

$$\begin{array}{c} M' \longrightarrow \\ \Big| \; \overset{\varphi_1 \, \varphi_3}{\underset{\varphi_2}{\Rightarrow}} \; M \\ \operatorname{Spec} B \otimes_A B \longrightarrow \operatorname{Spec} B \longrightarrow \operatorname{Spec} A \end{array}$$

"Spec $B \cap$ Spec B"

Say $(M', \varphi_1, \varphi_2, \varphi_3)$ the descent data. (or just (M', φ))

Proof:

$\alpha : M \to B \otimes M' \quad m \mapsto 1 \otimes m$

$\beta : M \to B \otimes M' \quad m \mapsto \varphi(m \otimes 1)$

$$\begin{tikzcd} M' \otimes B \arrow[r, "\alpha\otimes 1 - \beta \otimes 1"] \arrow[d, "\varphi"] & B \otimes M' \otimes B \arrow[d, "\varphi_1"] \\ B \otimes M' \arrow[r, "p_1 - p_2"'] & B \otimes B \otimes M' \end{tikzcd}$$

By $\varphi_1 \circ \varphi_3 = \varphi_2$, the diagram commutes.

By faithful flatness, kernel of $\alpha \otimes 1 - \beta \otimes 1 = M \otimes_A B$ where $M = \{ m' \in M \mid 1 \otimes m - \varphi(m \otimes 1) = 0 \}$.

Kernel of $p_1 - p_2$ is M'. 5-lemma says $M \otimes_A B \cong M'$.

Theorem: Suppose $F: \operatorname{Spec} B \to S$ is a covering in $(\mathrm{Aff}/S)_{fppf}$.

Let $(M', \varphi_1, \varphi_2, \varphi_3)$ Descent data for $\tilde{M}'$ on Spec B.

Then $\exists!$ qcoh sheaf N on S s.t. $F^* N = \tilde{M}'$.

Corollary: $(\mathrm{QCoh})^{pf}$, Coh, Vect all have fppf descent.

Affine morphisms, closed subschemes satisfy descent.

Corollary: Polarized schemes have descent.

$$\begin{array}{l} \mathcal{P}ol \\ \downarrow \\ (Aff/S)_{fppf} \end{array}$$

obs $(f: X \to U, L)$ — someone asked what this — audience should look it up.

U affine L f-relatively ample.

f proper.

mors $(f, L) \longrightarrow (f', L')$ are

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ U' \arrow[r, "u'"'] & U \end{tikzcd}$$

$+$ an iso $u^* L \xrightarrow[\varepsilon]{\sim} L'$.

It's fibred category $(Aff/S)_{fppf}$, by

$$(f, L) \longmapsto U.$$

Suppose $U' \to U$ an fppf covering in $(Aff/S)_{fppf}$.

By proper flat base change $u'^* f_* L^{\otimes N} \cong f'_* u^* L^{\otimes N} \cong f'_* L'^{\otimes N}$.

$\Rightarrow$ $u'^* E = E'$

$\Rightarrow$ projective embedding base changes i.e.

$$X' \hookrightarrow \mathbb{P}(E')$$

$$X \hookrightarrow \mathbb{P}(E)$$

descent follows from for loc frees & closed subschemes.

Corollary: $\mathcal{M}_g$, $g \neq 1$ and $\mathcal{A}_g$ satisfy descent (objects of $\mathcal{M}_g$ are pol curves pol by ω $g \geq 2$, ω^{-1} $g = 0$).

(objects of $\mathcal{A}_g$ are polarized by Θ-divisor).

Def: $F \xrightarrow{p} C$ be a cat fibred in groupoids, C site.

$x, y \in F(U)$, define $\mathrm{Isom}(x,y) : C/U \longrightarrow \mathrm{Set}$

$$V \xrightarrow{f} U \longmapsto \mathrm{Isom}(f^*x, f^*y).$$

this is a presheaf on C/U.

Def: a <u>stack in groupoids</u> is a CFG $F \to (\mathrm{Aff}/S)_{fppf}$ s.t.

(1) $\mathrm{Isom}(x,y)$ are sheaves $\forall x, y \in F(U)$, $\forall U \in (\mathrm{Aff}/S)_{fppf}$.

(2) Every descent data is effective.

i') Isomorphisms are separated

ii') Isomorphisms glue.

Expl: $F =$ cat of iso classes of line bundles i.e. $U \mapsto \mathrm{Pic}(U)$. This is <u>not</u> separated.

Expl: Most examples from before can be made into stacks.

1) $QCoh$, $QCoh^{pf}$, and Coh are all stacks.
i.e. $U \mapsto (-)(U)$.

2) $\mathcal{M}_g$ & $\mathcal{A}_g$ ($g \neq 1$ for $\mathcal{M}_g$).

3.) Bun_G, and relative vector bundles. Vect_X.

Stackification: There's a functor $(-)^a : CFG \longrightarrow \mathrm{Stacks}$ left adjoint to forgetful
satisfying universal property $\mathrm{HOM}_{CFG}(F, G) \xrightarrow{\sim} \mathrm{HOM}_{\mathrm{Stacks}}(F^a, G)$.
G a stack

Expl: quotient stack let X be a scheme, w/ G-action, G $\underbrace{\text{smooth}}_{\text{maybe not need.}}$ affine group scheme

$[X/G]$ objects over U = $Hom_S(U,X)$

morphisms = $\{g \in G \mid g \circ u = u'\}$.

The particular case $X = *$ is called the classifying stack BG of G.