Motivation for stacks

Question: How to describe all objects A in a nice way? (Is it a variety / scheme / ...)

Ex. Describe all genus 0 curves? $M = \{\mathbb{P}^1_{\mathbb{C}}\}$ if over $\mathbb{C}$. More data! - over $\mathbb{R}$

- automorphisms

Def. $F : Sch^{op} \to Sets$ functor is repre'able if $F \simeq Hom_{Sch}(-, X)$ for some X.

Ex. $F(X) = \mathcal{O}_X(X)$ Then $F \simeq Hom(-, \mathbb{A}^1_{\mathbb{C}})$

Ex. $F_n(X) = \{$ all line bundles $\mathcal{L}$ on X + $(s_0, \dots, s_n)$ generate $\mathcal{L}$ globally$\}$

$= \{$ maps to $\mathbb{P}^n_{\mathbb{C}} \} = Hom(-, \mathbb{P}^n_{\mathbb{C}})$

Ex. $Varieties/\mathbb{C} \xrightarrow[F]{} Set$

$F(X) = \{$tangent spaces at $x \in X \mid x \in X\}$

$= Hom(Spec\ \mathbb{C}[\varepsilon]/(\varepsilon^2), -)^{\vee}$

Fact $F \simeq Hom(-, X)$ then $\exists$ bijection

$F(U) \leftrightarrow Hom(U, X)$

$\rightsquigarrow F(X) \leftrightarrow Hom(X, X)$

$\mathcal{U}_F \in F(X) \leftrightarrow id_X \in Hom(X, X)$

universal family

Ex. "degree d curves in $\mathbb{P}^2_{\mathbb{C}}$"

$\updownarrow$

$$\sum_{i+j+k=d} a_{ijk}\, x^i y^j z^k = 0 \qquad \mathbb{P}_{\mathbb{C}}^{\binom{d+2}{2}-1} = \mathbb{P}^N \ni [a_{ijk}]$$

$\mathcal{U}_F$ is a subvariety of $\mathbb{P}_{\mathbb{C}}^N \times \mathbb{P}_{\mathbb{C}}^2$ given by $\sum a_{ijk} x^i y^j z^k = 0$

$F(Y) = \mathrm{Hom}(Y, \mathbb{P}_{\mathbb{C}}^N)$

= the set of pullback diagrams

$$\begin{tikzcd} \mathcal{U}_F \times Y \arrow[r] \arrow[d] & \mathcal{U}_F \arrow[d] \\ Y \arrow[r] & \mathbb{P}_{\mathbb{C}}^N \end{tikzcd}$$

Lemma $F : \mathrm{Sch}^{op} \to \mathrm{Sets}$

(1) Suppose $\exists$ family $\mathcal{E}$ (i.e. $\mathcal{E} \in F(S)$ for some S) s.t. $\mathcal{E}_s$ are all equal ($\mathcal{E}_s$: image of $\mathcal{E}$ under $F(S) \to F(\mathrm{Spec}\,\mathbb{C})$)

(2) $\mathcal{E}$ is not trivial i.e. $\mathcal{E}$ is not in the image of $F(\mathrm{Spec}\,\mathbb{C}) \to F(S)$

Then F is not representable.

Pf: Suppose not $F \simeq \mathrm{Hom}(-, X)$

$\mathcal{E} \in F(S) \leftrightarrow S \to X$ in $\mathrm{Hom}(S, X)$

$\mathcal{E} \mapsto \mathcal{E}_s$

$F(S) \to F(\mathrm{Spec}\,\mathbb{C})$ $\Rightarrow$ $\mathrm{Spec}\,\mathbb{C} \to S \to X$

always hit the same pt x_0

$\Rightarrow S \to \{x_0\} \in X$

$\Rightarrow F(\{x_0\}) \to F(S)$ Contradicting (2)

Ex. (of non repre'able functors)

- $\mathrm{Bun}_{r,d}(-)$ $\qquad$ - $\mathcal{M}_g$ $\quad g \geq 2.$
- Comm Rings $\to$ Sets
  $R \to$ Nilpotents of R.
- $\mathrm{Bun}_G(-)$

$\mathcal{M}_g$ is a functor

$F : \mathrm{Sch}/\mathbb{C} \to \mathrm{Set}$

$\mathrm{Hom}(-, \mathcal{M}_g)$ $\quad S \to \{ \mathcal{C} \to S \mid \mathcal{C}$ scheme $\forall s$ geo point $\mathcal{C}_s$ are genus g curves connected & nonsingular $\mathcal{C} \to S$ smooth flat$\}$

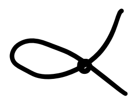
$$\begin{tikzcd}
 & \text{Sets} \arrow[dl] \arrow[dr] & \\
\text{Top Spaces} \arrow[d] & & \text{Presh on Sch} \arrow[d] \\
\text{Ringed spaces} \arrow[d] & & \text{Shv on } \mathrm{Sch}_{\acute{e}t} \arrow[d] \\
\text{Sch} \arrow[rr, hook] & & \text{Algebraic spaces}^{\acute{e}t} \arrow[ddddll, bend left] \\
 & \text{Groupoids} \arrow[d] & \\
 & \text{Prestacks} \arrow[d] & \\
 & \text{Stacks} \arrow[dl] \arrow[dr] & \\
\text{DM stacks}^{\acute{e}t} \arrow[rr, hook] & & \text{Alg. stacks}^{\acute{e}t}
\end{tikzcd}$$

$F : \mathrm{Sch}^{op} \to \mathrm{Sets}$

$F : \mathrm{Sch}^{op} \to \mathrm{Gpds}$

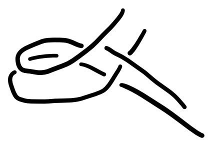
Étale sites

Ex. $y^2 - x^3 + x^2 = 0$



$\mathbb{C}[x,y] / (y^2 - x^3 + x^2)$

- more open sets



$\mathbb{C}[x,y,t] \Big/ (y^2 - x^3 + x^2,\ t^2 - (x-1))$

- GIT motivation: in étale case, for every points there's open sets containing the whole orbit
- Descent

name	objects	cover	Automorphism
Schemes	locally ringed space	by affine scheme	X
Alg. Spaces	Sheaf on ét sites	by affine scheme ét locally	finite groups
DM stacks	Stacks for étale site	in ét top.	finite groups
Alg. stacks	~	in smooth top	reductive groups