The Game of SET! (Solutions)

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The Madison Math Circle is an outreach organization seeking to show middle and high schoolers the fun and excitement of math! For more information about the Madison Math Circle as well as solutions to these exercises please visit our website at:

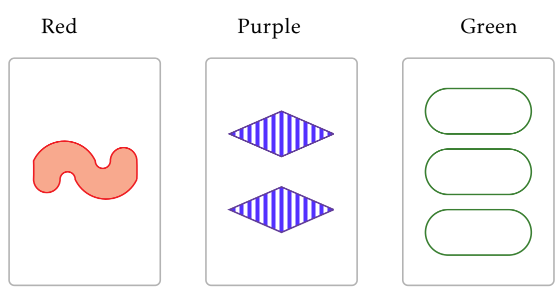
`math.wisc.edu/outreach/mathcircle`.

The Game:

Set is a card taking game, sorta similar to Concentration. It is played with a deck where each cards is labeled with a figure that differs in its:

- shape (diamond, oval, or squiggle),
- color (red, green, or purple),
- number (one, two, or three),
- shading (empty, slashed, or filled-in).

For example, below are three set cards:



The goal of the game is to take the most number of "**Sets**" possible where a set consists of:

3 cards are a Set if the characteristic (shape, color, number, shading) is the same or distinct for each of the cards.

More precisely three cards form a **SET** if each of the following hold true:

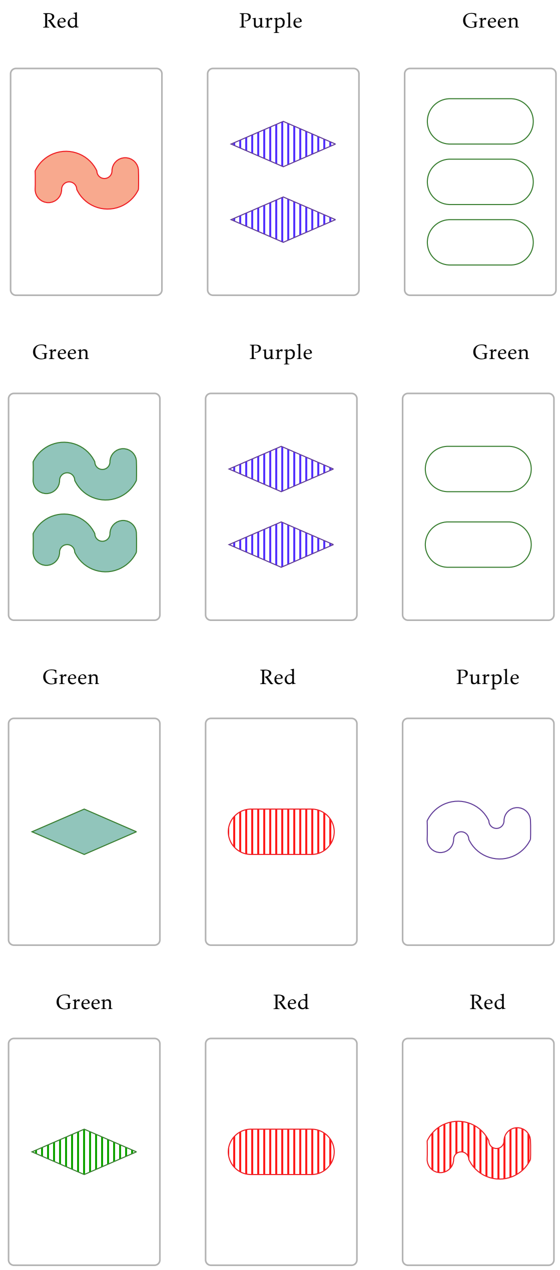
(1) all cards have the same shape **OR** all cards have different shapes,

(2) all cards have the same color **OR** all cards have different colors,

(3) all cards have the same number **OR** all cards have different numbers,

(4) all cards have the same shading **OR** all cards have different shading.

Exercise 1. Of the follows collections of cards precisely two are **Sets**, which are they?



Solution. The first and second rows are **Sets**. In the first row each card has a different color, a different, number, a different color, a different shape, and a different fill making it a **Set**. In the second row each card has the same number, a different color, different shape, and a different fill making it a **Set**. The remaining to rows are not **Sets** as they violate number (2) above. ☐

The Rules:

Other than the definition of a **Set** (make sure you have done Exercise **??**) the rules of Set are as follows:

- One player, designated the dealer, places 12 cards face up on the table.
- If a player sees three cards that form a **Set** they say "Set!" and grab the three cards.
- The dealer adds more cards to the table as they are taken away.
- If after a few minutes no one has found a **Set** the dealer adds three more cards; repeating until someone finds a **Set**.
- The game ends when all the cards have been dealt and no one can find any more **Sets**. The player with the most **Sets** at the end of the game wins!

Appetizers:

Exercise 2. Play a few games of Set!

Exercise 3. Each combination of shape, color, number, and shading appears exactly once in a Set deck. Does this tell you enough to know how many cards there are in the deck? If so how many? (Hint: If you are stuck think about how many different cards there would be if we only considered cards that had one object and are filled-in.)

Solution. Since I have told you exactly what appears on each card this is enough information to determine the size of the deck. In particular, the multiplication rule says that there are

$$(\#\text{ colors})\cdot\#\text{ shapes})\cdot\#\text{ numbers})\cdot\#\text{ fills}) = 3\cdot 3\cdots 3\cdot 3 = 3^4 = 81$$

different cards in the deck. One way to see why the multiplication rule works in this instance is by making a tree diagraming the choices so that first level of branches represent color, the second level represents shapes, the third represents number, and the fourth represents fill. □

Exercise 4. If you draw two cards from the Set deck how many cards remain in the deck such that they form a **Set** with the first two cards? (Hint: Remember each combination of shape, color, number, and shading appears exactly once.)

Solution. Given two cards A and B there is precisely on other card in the set C such that A, B, C forms a **Set**. In order to see this note that given two cards A and B there colors are either the same or different. If A and B have the same color then for A, B, C to be a set C must have the same color. On the other hand, if A and B are different colors C must be a color different from A and B so that A, B, C forms a set. Hence in either scenario the color of C is determined. Put differently, given that A, B, and C form a **Set** we are able to complete the following table (I have filled out the first half for you):

Color of A	**Color of B**	**Color of C**
	Red	Red
Red	Green	Purple
	Purple	Green
	Red	Purple
Green	Green	Green
	Purple	
	Red	
Purple	Green	
	Purple	

Replacing the word "color" with the other properties (shape, number, shading) shows that each property of C is determined by the properties of A and B implying there is only one such C in the deck. □

Exercise 5. If you randomly draw three cards from the Set deck what is the probability they form a **Set**?

Solution. This can be tricky, but using our solution to the previous exercise we can make a slick argument to show the probability of picking a **Set** is 1/79 as follows: Pick two cards, call them A and B, from the deck so that $81 - 2 = 79$ cards remain. By Exercise 4 there is precisely one card, call it C, such that A, B, C form a **Set**. Thus, the probability that I get a **Set** is the probability of picking C out of the remaining 79 cards i.e. 1/79. □

Exercise 6. How many different **Sets** can be formed?

Solution. The key to this solution is again Exercise 4. In particular, if we pick two cards A and B there is precisely one remaining card C such that A, B, C forms a **Set**. Therefore, the multiplication principal says the number of different ways to pick a **Set**, drawing one card at a time, is given by:

$$(\text{\# of choices for card } A) \cdot (\text{\# of choices for card } B) \cdot (\text{\# of choices for card } c) = 81 \cdot 80 \cdot 1 = 6480.$$

However, this is *not* the number of different **Sets** in the deck because in this count we kept track of the order in which we drew the cards. Put differently we counted **Sets** together with ways to label the three cards A, B, C. So in order to count just the distinct **Sets** we must take into account the number of ways we can label three cards with A, B, and C. By the multiplication rule there are 6 different ways to order the letters A, B, C:

$$ABC, ACB, BAC, BCA, CAB, CBA$$

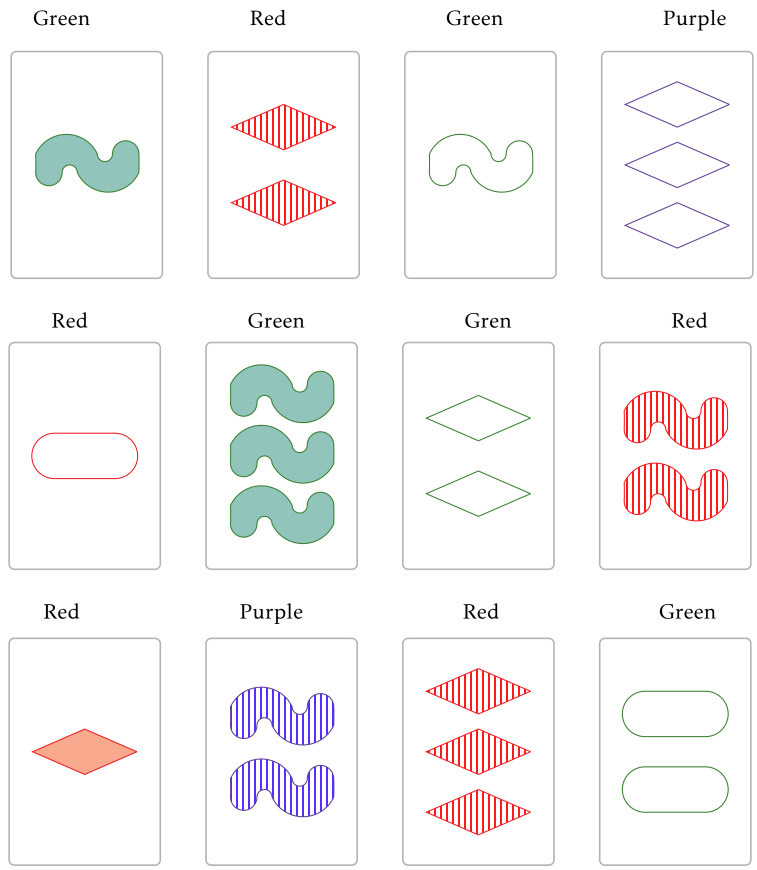
meaning that the number of distinct **Sets** in a Set deck is:

$$\frac{81 \cdot 80}{3!} = \frac{81 \cdot 80}{6} = \frac{6480}{6} = 1080.$$

□

Main Course:

The game of Set begins with 12 cards being placed on the table, however, it is possible for there to be *no* **Sets** amongst the 12 cards. For example, the 12 cards laid out below contain no **Sets**:

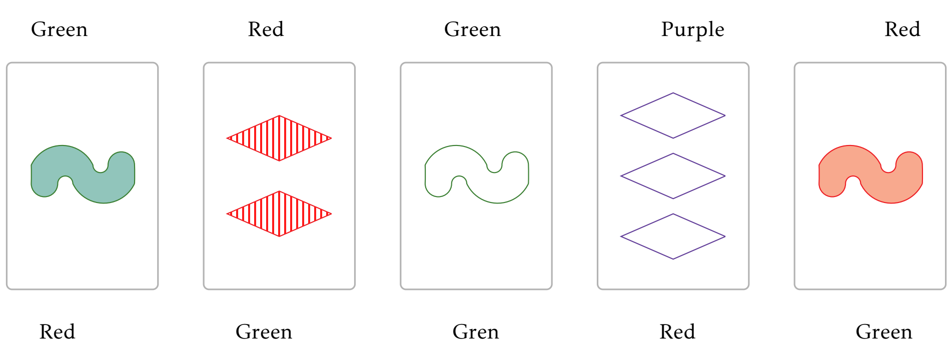


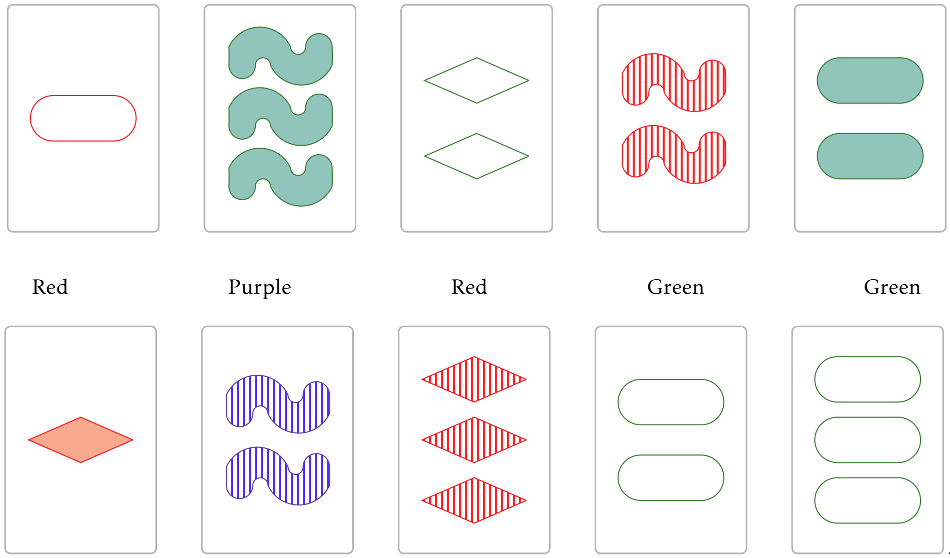
Take a moment to convince yourself this is true. When this happens the dealer places three more cards on the table; repeating until there is a **Set**. The goal of these exercise is to explore the question:

Question 1. What is the most number of times the dealer will have to add cards before we can guarantee a **Set** exists amongst the dealt cards? (i.e. what is most number of cards that can be on the table before they must contain a **Set**?)

Exercise 7. Show it is possible for there to be 15 cards on the table without any **Sets** present. (Hint: Try adding three cards to the 12 card example given above.)

Solution. Solutions will vary. One example of such a collection of cards is:

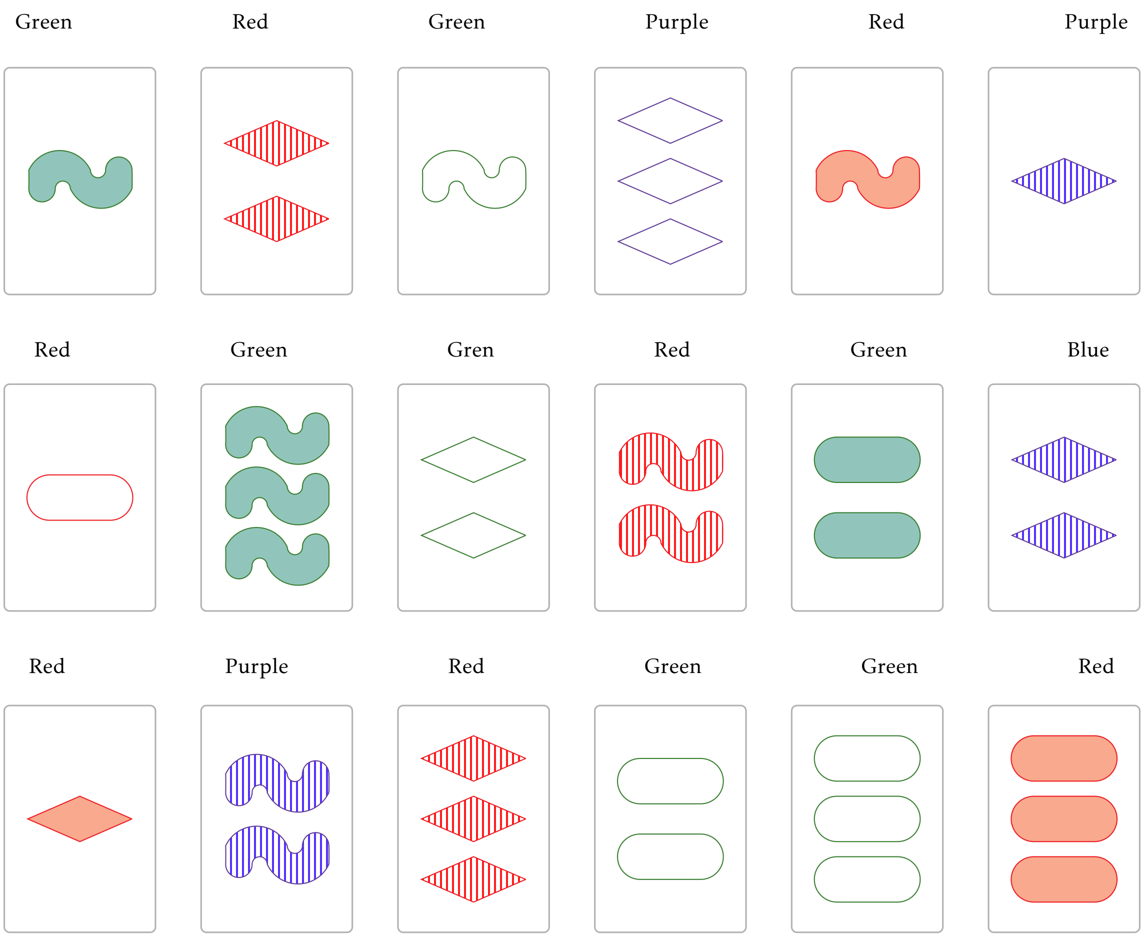




□

Exercise 8. Show it is possible for there to be 18 cards on the table without any **Sets** present. (Hint: Try adding three cards to the 15 card example given you found in the previous exercise.)

Solution. Solutions will vary. One example of such a collection of cards is:



□

At this stage if you have done Exercises **??** and **??** we know it is possible for 18 cards to not contain a **Set**. What if we add three more cards? Think for a moment whether you add three more cards to your example from Exercise **??** so that the 21 cards still contain no **Set**. (Don't think too long...)

Hopefully, you weren't able to come up with an example of 21 cards containing no **Set**. (If you did go back and double check your work.) However, just because it is hard to find such an example does not mean such an example does not exists. In math speak one would say, "Proof by examples is not a proof."

In particular, if we wanted to *prove* there are no such examples with 21 cards we would need to check *every* possible collection of 21 cards. There are:

$$13636219405675529520 \approx 1.4 \times 10^{19}$$

such collections... So we definitely are not going to be showing this checking case by case. It turns out that this is true, every collection of 21 cards contains at least one **Set**! However, proving this turns out to be surprisingly tricky!

A Late Night Snack

In the previous exercise we focused our attention on playing Set with cards that have four characteristics (shape, color, number, shading), but we could actually envision such a game where our cards have n characteristics (with three options for each) for any natural number n. We call such a game n-Set. So 4-Set is just regular old Set. It turns out that n-Set is both very complicated and very interesting mathematically. The University of Wisconsin's very own Jordan Ellenberg (with his co-authors) recently made huge advancements in understanding n-Set! See the Quanta article *Simple Set Game Proof Stuns Mathematicians* (available on-line) for more info about these breakthroughs.