Schlessinger's Criterion

Kevin Dao

August 1, 2025

WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

Definitions

Definition

A deformation functor $D : \mathrm{Art}^{loc}/k \to \mathrm{Set}$ is **pro-representable** if there exists a complete noetherian local k-algebra R with residue field k and an isomorphism

$$h_R := \mathrm{Hom}_{\mathrm{Art}^{loc}/k}(R, -) \xrightarrow{\sim} D.$$

Definition

A morphism of deformation functors $\alpha : F \to G$ is **smooth** if $F(B) \to F(A) \times_{G(A)} G(B)$ is surjective for all small extensions $B \to A$.
(For emphasis I might call this formally smooth).

Definition

The pair (R, α) where $\alpha : h_R \to D$ and R is a complete Noetherian local k-algebra is a **hull for** D if $h_R(k[\epsilon]/\epsilon^2) \to D(k[\epsilon]/\epsilon^2)$ is an isomorphism and α is smooth.

Note: Prorepresentable using $R \implies R$ is a hull.
Remark: Two hulls are noncanonically isomorphic. While the R for pro-representable is unique up to unique isomorphism.
Remark II: If global functor is representable, then its deformation functors are prorepresentable. OTOH, if one is considering algebraic stacks, then the definition of a hull shows up more naturally.

Definitions II

Remark

If $\alpha : F \to G$ *is formally smooth, then one can induct on length (if* B *is Artinian local) to conclude that* $F(B) \to G(B)$ *is always surjective.*

Baby Schlessinger

Theorem ("Baby Schlessinger")

(1) A hull for D exists iff D admits a tangent-obstruction theory.
(2) D is prorepresentable iff $(T^1 \otimes M)$ acts simply transitively on the set of lifts aka the exact sequence from last time was left exact

$$0 \to T^1 \otimes M \to D(B) \to D(A) \to T^2 \otimes M.$$

Why is Baby Schlessinger true?

Remark

What does the existence of a hull for G have to do with anything?
If a hull exists for G, call it S with $\alpha : h_S \to G$, then G admits a tangent-obstruction theory. One can take $T^1 = (m_S/m_S^2)^\vee$ one considers the diagram

$$\begin{array}{ccccccc} T_S^1 \otimes M & \longrightarrow & h_S(B) & \longrightarrow & h_S(A) & \longrightarrow & T_S^2 \otimes M \\ \downarrow & & \downarrow & & \downarrow & & \\ T_G^1 \otimes M & \longrightarrow & G(B) & \longrightarrow & G(A) & & \end{array}$$

What should the candidate for T_G^2 be? I think the choice is to pick $T_G^2 := T_S^2$ because by smoothness, lifting $\xi \in G(A)$ to $G(B)$ is equivalent to being able to lift the corresponding $\xi \in h_S(A)$ to $\xi \in h_S(B)$.
Conversely, let (T^1, T^2) be a tangent-obstruction theory. The idea is to then build a hull. I think the proof of Schlessinger's criterion handles this because if we work over k, then baby Schlessinger's hypotheses $\implies$ Schlessinger's theorem's hypotheses.
Jeremy sketches this in his slides too.

Why is Baby Schlessinger true? II

Remark

Assume G is prorepresentable by R. So, R is a hull and $h_R(B) \to h_R(A) \times_{G(A)} G(B)$ is a bijection. This bijection implies the left exactness of

$$0 \to T^1_G \otimes M \to G(B) \to G(A) \to T^2_G \otimes M.$$

since we know it to be the case for h_R.
Now assume there is a tangent obstruction theory with left exactness. Find a hull R by the first part. Then I need $h_R(B) \to h_R(A) \times_{G(A)} G(B)$ to be a bijection for all small extensions $B \to A$. Left exactness implies $T^1_G \otimes M$ acts simply transitively on lifts from $G(A)$ to $G(B)$ and the bijection on tangent spaces gives identifies $T^1_G \otimes M$ with $T^1_R \otimes M$. But that gives the bijection via a diagram chase.

Schlessinger

$\mathcal{C} := Art^{loc}/k$

Can also be more general with $\mathcal{C} :=$ Artinian Λ-algebras with residue field k and Λ a complete noetherian local k-algebra.

Theorem (Schlessinger)

Let $F : \mathcal{C} \to \mathrm{Set}$ be a deformation functor. Let $R \to A$, $S \to A$ be two maps in $\mathcal{C}$. Consider the map

$$F(R \times_A S) \to F(R) \times_{F(A)} F(S) \qquad (\dagger)$$

Then F has a hull iff S1-S3 hold and F is prorepresentable iff S1-S4 hold.

- S1 *(gluing) if $R \to A$ is small, then the map $(\dagger)$ is surjective*
- S2 *(tangent spaces make sense) $(\dagger)$ is bijective for $R = k[\epsilon]/\epsilon^2$ and $A = k$,*
- S3 *(finite dim) $\dim_k F(k[\epsilon]/\epsilon^2) < \infty$ (as before, this is a k-vector space from previous talks...)*
- S4 *(separatedness) if $R \to A$ and $S \to A$ coincide, then $(\dagger)$ is a bijection.*

Question? For $S4$ we do not need to assume $R \to A$ is small. Can one weaken S4 so that we only need to check small extensions?

Hull $\implies$ S1-S3 and Pro-representable implies S1-S4

Assume $\alpha : h_{\widetilde{R}} \to F$ is a **hull** for F.
S1: Assume $R \to A$ is small and $S \to A$ is any map.

$$\begin{array}{ccccc}
h_{\widetilde{R}}(R \times_A S) & \xrightarrow{\simeq} & h_{\widetilde{R}}(R) \times_{h_{\widetilde{R}}(A)} h_{\widetilde{R}}(S) & \xrightarrow{\text{smooth}} & h_{\widetilde{R}}(A) \times_{F(A)} F(R) \times_{h_{\widetilde{R}}(A)} h_{\widetilde{R}}(S) \\
 & & & & \downarrow \simeq \\
\downarrow \alpha & & & & F(R) \times_{F(A)} h_{\widetilde{R}}(S) \\
 & & & & \downarrow \text{smooth} \\
F(R \times_A S) & \longrightarrow & & & F(R) \times_{F(A)} F(S)
\end{array}$$

Since the composition along $\neg$ is surjective, the bottom map is surjective.

Hull $\implies$ S1-S3 and Pro-Representable implies S1-S4

S2: If $A = k$, $R = k[\epsilon]/\epsilon^2$, then WTS $F(S[\epsilon]/\epsilon^2) \to T^1_F \times F(S)$ is bijective. Use tangent-obstruction to see this.

$$\begin{array}{ccccccccc} & & T^1_F \otimes S & \longrightarrow & F(S[\epsilon]/\epsilon^2) & \longrightarrow & F(S) & \longrightarrow & T^2_F \otimes S \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T^1_F \otimes k & \longrightarrow & F(k[\epsilon]/\epsilon^2) & \longrightarrow & F(k) & \longrightarrow & T^2_F \otimes k \end{array}$$

Now do a diagram chase to show that $F(S[\epsilon]/\epsilon^2)$ is actually the fibre product for the middle square.

S3: $\dim_k h_{\widetilde{R}}(k[\epsilon]/\epsilon^2) < \infty$ is clear since $\widetilde{R}$ is must have finite dimensional tangent space by assumption.

Now assume F is pro-representable i.e. there is an isomorphism $h_{\widetilde{R}} \to F$.

S4: We check that there is a bijection

$$h_{\widetilde{R}}(R \times_A R) \xrightarrow{\sim} F(R \times_A R) \to F(R) \times_{F(A)} F(R) \xleftarrow{\sim} h_{\widetilde{R}}(R) \times_{h_{\widetilde{R}}(A)} h_{\widetilde{R}}(R)$$

The desired bijection (the middle map) is then clear because we have a bijection $h_{\widetilde{R}}(R \times_A R) \to h_{\widetilde{R}}(R) \times_{h_{\widetilde{R}}(A)} h_{\widetilde{R}}(R)$.

The next 5 slides consist of the proof of Schlessinger's Theorem which I will skip in the actual talk.
One can find the proof in Schlessinger's "Functors of Artin Rings" paper.

Let's jump to the examples.

S1-S3 $\implies$ Hull

Idea: Build the hull an inverse limit. By Yoneda, the map $h_{\widetilde{R}} \to F$ corresponds to an element $\xi \in F(R)$ which we need to construct.

Step 1: We want $\widetilde{R}/\mathfrak{m} = k =: \widetilde{R}_1$.

Step 2: T_F is a k-vector space. Let $x_1, \ldots, x_r$ be a basis. Then set $S := \Lambda[[x_1, \ldots, x_r]]$ and define

$$\widetilde{R}_2 := \frac{S}{\mathfrak{m}_S^2 + \mathfrak{m}_\Lambda S} = \frac{\widetilde{R}}{\mathfrak{m}_{\widetilde{R}}^2 + \mathfrak{m}_\Lambda \widetilde{R}} \cong k[\epsilon]/\epsilon^2 \times_k \cdots \times_k k[\epsilon]/\epsilon^2$$

By S2, I have $F(\widetilde{R}_2) \cong F(\prod_1^r k[\epsilon]/\epsilon^2) = T_F \times \cdots \times T_F = T_F \otimes T_F^\vee$. So $\xi_2 := id_{T_F \otimes T_F^\vee} = \sum x_i \otimes x_i^\vee$.

Step 3: More generally, build $\widetilde{R}_q, \xi_q \in F(\widetilde{R}_q)$ with $\widetilde{R}_q = S/J_q$ such that (1) $\widetilde{R}_q/J_{q-1} = \widetilde{R}_{q-1}$, (2) $\xi_q \to \xi_{q-1}$ under $F(\widetilde{R}_q) \to F(\widetilde{R}_{q-1})$, (3) $\varprojlim_q (\widetilde{R}_q, \xi_q)$ is the desired hull, (4) $\varprojlim_q \xi_q : h_{\widetilde{R}} \to F$.

Achieving Step 3

Claim: Let J_q be the the minimal ideal J such that $\mathfrak{m}_S J \subseteq J \subseteq J_{q-1}$ **and** ξ_{q-1} lifts to $F(\widetilde{R}_q) \to F(\widetilde{R}_{q-1})$. It exists because if J, K satisfy this then $J \cap K$ also does. (Note J_q is a valid choice but might not be minimal). One uses H1 to show that $J \cap K$ also satisfies the lifting property.

Whats left? We need to check (1) $T_R \to T_F$ is an isomorphism and (2) $h_R \to F$ is smooth. But (1) is clear by Step 2.
To check (2), WTS $h_{\widetilde{R}}(B) \to h_{\widetilde{R}}(A) \times_{F(A)} F(B)$ is surjective for any small extension $0 \to M \to B \to A \to 0$.
First can reduce to small extensions with $\dim M = 1$ because if $B \to A \to Z$ is a composition of two small extensions, we can form

$$h_{\widetilde{R}}(B) \to h_{\widetilde{R}}(A) \times_{F(A)} F(B) \to h_{\widetilde{R}}(A') \times_{F(A')} F(A) \times_{F(A)} F(B) \cong h_{\widetilde{R}}(A') \times_{F(A')} F(B).$$

Achieving Step 3 continued

Note $B \times_A B \cong B \times_k k[\epsilon]/\epsilon^2$ via $(x, y) \to (x, x) \bmod \mathfrak{m}_B + y - x$. Now

$$F(B) \times T_F = F(B) \times_{F(k)} F(k[\epsilon]/\epsilon^2) \overset{\sim}{} \xrightarrow{\text{using S2}} F(B \times_k k[\epsilon]/\epsilon^2) \overset{\text{above}}{\cong} F(B \times_A B) \twoheadrightarrow F(B) \times_{F(A)} F(B).$$

Now if I chase through the maps, $(x, \delta) \to (x, \delta \cdot x)$ so that $F(B) \to F(A)$ is a T_F-torsor. Now let $f \in h_{\widetilde{R}}(A)$ and $\eta \in F(B)$ such that $\xi(f) = \bar{\eta} \in F(A)$. By transitivity of the action, we need to find *any* lift of f to $h_{\widetilde{R}}(B)$.

Achieving Step 3 continued again

Want to find any lift of f to $h_{\widetilde{R}}(B)$. Since $f : \widetilde{R} \to B$, I know f factors through some $\widetilde{R}_q$.

$$\begin{array}{ccccc} S & \xrightarrow{w} & \widetilde{R}_q \times_A B & \longrightarrow & B \\ \downarrow & \nearrow & \downarrow{\scriptstyle pr_1} & & \downarrow \\ \widetilde{R}_{q+1} & \longrightarrow & \widetilde{R}_q & \xrightarrow{f} & A \end{array}$$

Claim 1. Either pr_1 splits or w is surjective.
Assume pr_1 is not split. Consider $\mathrm{Im}(w)$ which is a subring. Now if w is not surjective, then $\mathrm{Im}(w)$ is a subring. It maps surjectively onto $\widetilde{R}_q$ along pr_1. So the kernel of $\mathrm{Im}(w) \to \widetilde{R}_q$ is properly contained in the kernel of pr_1 which is also 1-dimensional k-vector space. So that means the kernel is zero. But then I can form the section $\widetilde{R}_q \to \mathrm{Im}(w) \subseteq \widetilde{R}_q \times_A B$ which is a contradiction.
Claim 2. This gives a lift $\ell : \widetilde{R}_{q+1} \to \widetilde{R}_q \times_A B$ as follows. If pr_1 is split, use the section to get the lift.
Now assume w is surjective. By S1, $F(\widetilde{R}_q \times_A B) \to F(\widetilde{R}_q) \times_{F(A)} F(B)$ is surjective so I can lift $\xi_q \in F(\widetilde{R}_q)$ to $\widetilde{\xi_q} \in F(\widetilde{R}_q \times_A B)$. But by minimality of J_{q+1} and smallness, I get that $\mathfrak{m}_S J_q \subseteq \ker(w) \subseteq J_q$ and ξ_q lifts to $S/\ker(w)$ with $J_{q+1} \subseteq \ker(w)$.
Using this, I get a map $R_{q+1} \to B$ that lifts f.

Step 4

It remains to show that if S4 is true, then I get pro-representability. Clearly pro-representability implies S4.
Assume $F(R \times_A R) \to F(R) \times_{F(A)} F(R)$ is a bijection.
I claim $\widetilde{R}$ actually prorepresents F. It suffices to show $\xi : h_{\widetilde{R}}(B) \to F(B)$ is an isomorphism for all B.
We can prove this by induction on length (length zero being trivial).
Let $0 \to M \to B \to A \to 0$ with $\dim_k M = 1$. Form

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_{\widetilde{R}} \otimes M & \longrightarrow & h_{\widetilde{R}}(B) & \longrightarrow & h_{\widetilde{R}}(A) \\ & & \Big| \text{S2} & & \downarrow & & \downarrow \simeq\text{,inductive hyp} \\ 0 & \longrightarrow & T_F \otimes M & \longrightarrow & F(B) & \longrightarrow & F(A) \end{array}$$

Here, S4 is used to have left exactness.

Example

Remember that the conditions are (S1) (†) is surjective when $R \to A$ is small, (S2) (†) is bijective if $[R \to A] = [k[\epsilon]/\epsilon^2 \to k]$, (S3) tangent space is finite dimensional, and (S4) (†) is bijective if $[R \to A] = [S \to A]$.

Example

Let $P : \mathrm{Art}^{loc}/k \to \mathrm{Set}$ be given by $P(A) :=$ set of line bundles $\mathcal{L}_A$ on X_A which are flat deformation of $\mathcal{L}$ on X up to isomorphism.
Then this is prorepresentable with $T_P = H^1(X, O_X)$ if $h^1(X, O_X) < \infty$.
The $\widetilde{R}$ in this case is of course $k[[x_1, \dots, x_r]]$ where $x_1, \dots, x_r$ form a basis for T_P.
Prorepresentability follows from Grothendieck's Theorem on Picard Functor but we can do this using Sclessinger's Criterion.
- $P(k)$ is a single point so I have a deformation functor.
- S1 holds because if I take $\mathcal{L}'/X_R$ and $\mathcal{L}''/X_S$ both deforming $\mathcal{L}/X_A$, then I can form $\mathcal{L}' \times_{\mathcal{L}} \mathcal{L}''$ one $X \times_k (R \times_A S)$. (See next slide for statement on why.)
- S2 follows since deformations over $S[\epsilon]/\epsilon^2$ should correspond to deformations over S and $k[\epsilon]/\epsilon^2$.
- S3 is $\mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) = H^1(X, O_X)$
- S4 follows when $End(\mathcal{L}) = H^0(X, O_X) = k$ because iso class of deformations form a torsor under $H^1(M \otimes O_X)$ in that case. (Slight gap–this is for the case of small extensions $R \to A$ in S4. One would need to iterate to get it for all $R \to A$.)

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Example

Here are some examples. Remember that the conditions are (S1) (†) is surjective when $R \to A$ is small, (S2) (†) is bijective if $[R \to A] = [k[\epsilon]/\epsilon^2 \to k]$, (S3) tangent space is finite dimensional, and (S4) (†) is bijective if $[R \to A] = [S \to A]$.

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Example (technical fact)

One uses the following technical fact to justify "fibre product of sheaves does what one expects".

Lemma

Let R, S, A be rings with maps $R \to A, S \to A$. Let M_R, M_S, M_A be modules on the respectivel rings with maps $M_R \to M_A, M_S \to M_A$ of R-modules and S-modules. Assume $M_R \otimes_R A \to M_A$ and $N := M_S \otimes_S A \to M_A$ are isomorphisms.
(a) If $S \to A$ is surjective, then $N \otimes_{R \times_A S} R \to M_R$ is an isomorphism.
(b) If $\ker(S \to A)$ is square zero and M_R, M_S are R-flat and S-flat resp., then N is $(R \times_A S)$-flat and $N \otimes_{R \times_A S} S \to M_S$ is an isomorphism.

Remark: Apparently this result is due to Milnor and comes from Milnor's book on K-theory. The version above is taken from Hartshorne's Deformation Theory book.

Example II

Example (Example II)

Let $\mathcal{F}$ be a coherent sheaf on a projective scheme X. Let F be the functor with $F(A)$ the set of deformations $\mathcal{F}$ of $\mathcal{F}_0$ over A up to isomorphism (here we fix the isomorphism $\mathcal{F}' \times_A k \to \mathcal{F}$. Then F has a hull. However, S4 may fail.
The functor is prorepresentable (aka S4 holds) if we assume also that $\mathcal{F}$ is simple.
One expects S4 to fail without simplicity since I can imagine forming $\mathcal{F}_R \times_{\mathcal{F}_A} \mathcal{F}_R$ but now using a nontrivial automorphism $\phi : \mathcal{F}_A \to \mathcal{F}_A$.

Example III

Example

We know the Hilbert scheme exists so the associated deformation functors are prorepresentable.
Exercise: Use Schlessinger's criterion to check that the local Hilbert functors are prorepresentable.

Example

Let X_0/k be a scheme. Then deformations of X_0 over local Artin rings has a hull iff either one holds (a) X_0/k has isolated singularities or (b) X_0/k is projective.
If $H^0(T_{X_0}) = 0$, then the functor is actually pro-representable.

The Dimension of R

Theorem

Let (T^1, T^2) *be a tangent-obstruction theory for* F*. Then if* R *is a hull for* F*, we have* $\dim R \geq \dim T^1 - \dim T^2$*.*

Proof of Theorem Regarding Dimension of R

Lemma

Let $R \in \mathrm{Loc}_k$. Let $S := k[[x_1, \ldots, x_r]] \twoheadrightarrow R$ with $T_R \cong T_S$ and J its kernel. Set $T^1 := (\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee$ and $T^2 := (J/\mathfrak{m}_S J)^\vee$. If $T^{i'}$ is another tangent-obstruction theory for R, then (a) $T^1 \cong T^{1'}$ and (b) there is a functorial injection $T^2 \hookrightarrow T^{2'}$.

For the theorem, we know $\dim R \geq \dim S -$ (minimal number of generators of J) and $\dim S = \dim T^1$. Now reduce J mod $\mathfrak{m}_S$ to get there are at least $\dim T^2$ generators. So what's left is to prove the lemma.

Proof of the Lemma

Part (a) that $T^1 \cong T^{1'}$ for any pair of tangent-obstruction theories was explained by Jeremy last time.
Part (b) requires work. I want to find some element $\eta \in \mathrm{Hom}(T^2, T^{2'})$. That isn't the hard part. The functorial injection is what makes it trickier.
First, apply the Artin-Rees Lemma to pick $i > 0$ such that $\mathfrak{m}_S^i \cap J \subseteq \mathfrak{m}_S J$.
Consider

$$M := \frac{(J + \mathfrak{m}_S^i)}{(\mathfrak{m}_S J + \mathfrak{m}_S^i)} = \frac{J}{\mathfrak{m}_S J} \qquad \& \qquad B := \frac{S}{\mathfrak{m}_S J + \mathfrak{m}_S^i}$$

Then,

$$0 \to M \to B \to A := B/M = \frac{S}{J + \mathfrak{m}_S^i} = \frac{R}{\mathfrak{m}_S^i R} \to 0$$

is a small extension.

Using the tangent obstruction theory $(T^{1'}, T^{2'})$,

$$h_R(B) \to h_R(R/\mathfrak{m}_S^i R) \xrightarrow{ob} T_2' \otimes M \overset{\text{Def of } M}{\cong} T^{2'} \otimes (T^2) \vee .$$

Now the image of $\pi : R \to R/\mathfrak{m}_S^i R$ is $ob(\pi)$ which gives a map $T^2 \to T^{2'}$.

Claim: $ob(\pi)$ is injective.

Suppose $ob(\pi) : T^2 = M^\vee \to T^{2'}$ failed to be injective. Let $(M/V)^\vee \subseteq M^\vee$ be its nontrivial kernel and form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & S & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \pi & & \\
0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \Downarrow = & & \\
0 & \longrightarrow & M/V & \longrightarrow & B/V & \longrightarrow & R/\mathfrak{m}_S^i R & \longrightarrow & 0
\end{array}$$

Proof of the Lemma Slide III

Now we get

$$\begin{array}{ccccc} h_R(B) & \longrightarrow & h_R(R/m_S^iR) & \xrightarrow{\pi \to ob(\pi)} & T^{2'} \otimes M \\ \downarrow & & \Downarrow = & & \downarrow ob(\pi)\to 0 \\ h_R(B/V) & \longrightarrow & h_R(R/m_S^iR) & \longrightarrow & T^{2'} \otimes (M/V) \end{array}$$

Now $ob(\pi)$ is the obstruction to existence of lift $\ell : R \to B$. Now, there is no obstruction in lifting $\ell' : R \to B/V$ according to the diagram above.
But the obstruction according to T^2 is given by the quotient map $M \to M/V$:

$$h_R(B/V) \to h_R(R/\mathfrak{m}_S^iR) \to T^2 \otimes (M/V) = M^\vee \otimes (M/V) = \mathrm{Hom}(M, M/V).$$

It is a quotient map so it is nonzero. But this contradicts the fact the diagram is commutative and $ob(\pi) \to 0$, and obstruction is independent of obstruction theory.