Sketchy Notes: Typos are to be expected and all mistakes/inaccuracies/idiocy are the sole responsibility of the person who wrote these notes. Nothing here is original.

0.1 Dimension and Smoothness of Bun_G**.** Our goal is to explain the proof of the following theorem. Throughout, X is a projective algebraic curve over an algebraically closed field $k = \bar{k}$ of genus $g > 1$ and G is a reductive algebraic group.

Theorem 0.1. The algebraic stack Bun_G of principle G-bundles on X is smooth and of dimension $\dim(G)(g-1)$.

First, let me summarize the relevant definitions used in the theorem.

Definition 0.2. An algebraic stack $\mathcal{X}$ is **smooth** if there exists a smooth atlas $U \to \mathcal{X}$ (i.e. the morphism $U \to \mathcal{X}$) is smooth and surjective) in which U is also a smooth scheme.

As for the dimension of an algebraic stack, there are many. However, since we know that Bun_G is supposed to be a very nice stack (i.e. smooth algebraic and locally of finite type) it helps to use a less general definition which we introduce later.

Second, I shall summarize the deformation theory that will be involved in proving the theorem.

Definition 0.3. Let $\mathcal{X}/\operatorname{Spec}(k)$ be an algebraic stack. We say $\mathcal{X}$ is **formally smooth** if for all

$$\begin{array}{ccc} T & \longrightarrow & \mathcal{X} \\ {\scriptstyle i}\downarrow & \overset{\exists}{\nearrow} & \downarrow \\ T' & \longrightarrow & \operatorname{Spec}(k) \end{array}$$

first order thickenings of T, there exists the map $T' \to \mathcal{X}$. Note that the lift does not need to be unique. With more assumptions on $\mathcal{X}/k$, one can make stronger assumptions on T.

Proposition 0.4 (Alper, Theorem 3.7.1)**.** Assume $f : \mathcal{X} \to \mathcal{Y}$ is a locally of finite type morphism between locally noetherian algebraic stacks with qcqs diagonals. Consider the 2-commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(A_0) & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow {\scriptstyle f} \\ \operatorname{Spec}(A) & \longrightarrow & \mathcal{Y} \end{array}$$

of solid arrows where $A \to A_0$ is a surjection of artinian local rings with residue field k such that $\ker(A \to A_0) \cong k$ (aka we consider a first order infinitesimal thickening). Then f is smooth if and only if there exists a lifting of every diagram as pictured above.

Remark 0.5. Let me try to describe why this is what we want. If $\mathcal{X} \to \operatorname{Spec}(k)$ is smooth, then the above lifting criterion holds. Then let $U \to \mathcal{X}$ be an atlas. The goal is to show that U is a smooth scheme. But $U \to \mathcal{X} \to \operatorname{Spec}(k)$ is a composition of smooth morphisms and so $U \to \operatorname{Spec}(k)$ is smooth. OTOH, if there is a smooth atlas $U \to \mathcal{X}$, and $\operatorname{Spec}(A_0) \to \mathcal{X}$ is given, then I can find a lift $\operatorname{Spec}(A) \to U$ by the proposition and the composition $\operatorname{Spec}(A) \to U \to \mathcal{X}$ is the desired lift of $\operatorname{Spec}(A_0) \to \mathcal{X}$.

Proposition 0.6. Let $\mathcal{X}$ be a locally of finite type and locally Noetherian algebraic stack over $\operatorname{Spec}(k)$ with qcqs diagonal. Then it is smooth if and only if it is formally smooth.

Proposition 0.7. Let X_0 be a scheme over k and $\mathcal{E}_0$ a vector bundle on X_0. Let X be a deformation of X_0 over some Artin local ring R i.e. X_0/R such that $X_0 \times_k R \cong X$. Then there exists a deformation $\mathcal{E}$ of $\mathcal{E}_0$ over R (i.e. $\mathcal{E}/R$ is a vector bundle such that $\mathcal{E} \otimes_R k \cong \mathcal{E}_0$) iff the obstruction class $o(\mathcal{E}_0) \in H^2(X_0, End_{O_{X_0}}(\mathcal{E}_0) \otimes J)$ vanishes where J is the kernel $\ker(R \to k)$.

Proposition 0.8 (Upgraded Version)**.** In the previous proposotion, instead of working with $R \to k$, one can work instead with $R \to R'$ where R, R' are both Artinian local rings with residue fields k.

0.3 Smoothness as Vanishing of Obstruction. The idea of this proposition was discussed 2 weeks ago. It is especially helpful here that we are working with vector bundles.
Remark: See Theorem 7.1 of Hartshorne's Deformation Theory.

Now let us apply this proposition to prove that Bun_G is smooth. One can observe we do not need the full power of the proposition.

Consider the diagram

$$\begin{array}{ccc} \mathrm{Spec}(T) & \longrightarrow & \mathrm{Bun}_G \\ \big\downarrow i & \nearrow & \\ \mathrm{Spec}(T') & & \end{array}$$

Now assume $\mathrm{Spec}(T) \to \mathrm{Bun}_G$ is given by a family $\mathcal{E}_T$ of G-bundles over the scheme T......then the obstruction class is in H^2 which vanishes.

0.4 The Tangent Stack. There are notes online by Raskin which discuss the cotangent stack and the material below – it might be a better read than what I have written here below. Let $\pi : U \to \mathcal{X}$ be a fixed atlas with U smooth (here we are assuming $\mathcal{X}$ is smooth). Then the tangent sheaf T_X is a vector bundle and for any test scheme S with map $f : S \to \mathcal{X}$, we can consider the diagram

$$\begin{array}{ccc} f^*T_U & \longrightarrow & T_U \\ \big\downarrow & & \big\downarrow \\ S \times_{f,\mathcal{X},\pi} U & \longrightarrow & U \\ \big\downarrow h & & \big\downarrow \pi \\ S & \xrightarrow{f} & \mathcal{X} \end{array}$$

Now recall that for any smooth surjective map of schemes $\rho : Y \to Z$, there is an exact sequence of sheaves over Y

$$0 \to T_{Y/Z} \to T_Y \to \rho^* T_Z \to 0$$

which we can regard as saying there is a quasiisomorphism

$$(T_{Y/Z} \to T_Y) \simeq \rho^* T_Z.$$

This will be our tool for defining the cotangent stack. To describe the tangent bundle on $Z = \mathcal{X}$, it should be the case that we try and describe some two-term complex

$$T_{U/\mathcal{X}} \to T_U.$$

Using the above diagram, we can form the exact sequence

$$0 \to T_{S \times_{f,\mathcal{X},\pi} U/S} \to T_S \to h^* T_S \to 0.$$

The collection of $T_{S\times_{f,\mathcal{X},\pi}U/S}$ with S varying over some covering to $\mathcal{X}$ glue together to give me a vector bundle which I call $T_{U/\mathcal{X}}$. Varying the S over some covering, the T_S glue together to give me T_U. The collection of morphisms then descend to a morphism $T_{U/\mathcal{X}} \to T_U$.

Now, we remark that there is nothing special about choosing the atlas U. We could have replaced U by some covering of U and hence, a covering of $\mathcal{X}$. So altogether, $T_{U/\mathcal{X}}$ makes sense for any $U \to \mathcal{X}$ in the smooth site over $\mathcal{X}$.

Definition 0.9. The **tangent stack** of $\mathcal{X}$ is the functor that associates to each $U \to \mathcal{X}$ in the smooth site to the two-term complex

$$T_{U/\mathcal{X}} \to T_U$$

of quasicoherent sheaves over U concentrated in degrees -1 and 0. That this gives a functor valued in groupoids is explained in the next paragraph.

To each two-term complex $\cdots \to A^{-1} \to A^0 \to \dots$ one can form a groupoid h^1/h^0 as follows. Its objects are elements of A^0 and the morphisms $x \tilde{y}$ are given by those $v \in A^{-1}$ such that $dv = y - x$ where $v \in A^{-1}$. So for example, an $x \in A^0$ has an automorphism for every $v \in A^{-1}$ with the property $dv = x - x = 0$.

Definition 0.10. If $S \to \mathcal{X}$ is a S-smooth point of $\mathcal{X}$, then set $\dim_S(\mathcal{X}) := \operatorname{rank}(T_S) - \operatorname{rank}(T_{S/\mathcal{X}})$. When $\mathcal{X}$ is a smooth algebraic stack, $\dim_S(\mathcal{X})$ is independent of S and we denote this quantity by $\dim(\mathcal{X})$. For our later computations, we will take $S := \operatorname{Spec}(k)$ and let P denote the principle G-bundle on X associated to $S \to \operatorname{Bun}_G$.

0.5 Dimension of Bun_G**.** There are notes online from a talk by Victor Ginzburg on this which are better than the notes here if one wants more details.

Theorem 0.11. $T_P(\operatorname{Bun}_G) \cong H^1(X, \mathfrak{g}_P)$ while automorphisms of $T_P(\operatorname{Bun}_G)$ at P are determined by $H^0(X, \mathfrak{g}_P)$.

Proof. We start with the Tannakian formalism for describing principle G-bundles.

Claim. Let G be an affine algebraic group. The data of a G-bundle on X is equivalent to the data of a monoidal exact functor $\operatorname{Rep}(G) \to \operatorname{Vect}(X)$ where $\operatorname{Rep}(G)$ is the tensor category of *finite* dimensional representations of G and $\operatorname{Vect}(X)$ is the tensor category of vector bundles on X.

Proof of claim. Given a principle G-bundle P on X, we can define a functor F_P : $\operatorname{Reg}(G) \to \operatorname{Vect}(X)$ via $F_P(V) := P \times_G V$. This is monoidal because one can check the isomorphism

$$P \times_G (V \otimes W) \cong (P \times_G V) \otimes (P \times_G W).$$

locally. The functor is also exact because the associated vector bundle construction is exact.

Let $F : \operatorname{Rep}(G) \to \operatorname{Vect}(X)$ be a monoidal exact functor. Apply F to[1] $\operatorname{Reg}(G) \otimes \operatorname{Reg}(G) \to \operatorname{Reg}(G)$ to get a commutative algebra structure on $F(\operatorname{Reg}(G))$. Then take $P := \operatorname{Spec}_X(F(\operatorname{Reg}(G)))$. ☐

[1]**Objection!** F is only defined for finite dimensional representations o what gives? Yes, but the extension of F to all representations is uniquely determined. This has to do with the fact that any quasicoherent sheaf is an inductive limit of coherent sheaf and representations can be viewed as quasicoherent sheaves on BG with the coherent sheaves corresponding to those with finite dimension.

From this theorem, we know that the tangent complex of Bun_G at a point P is identified with the two -term complex

$$H^0(\mathfrak{g}_P) \xrightarrow{0} H^1(\mathfrak{g}_P).$$

The dimension of Bun_G is equal to the Euler characteristic of this complex (concentrated in degrees $-1, 0$) which is $-\chi(\mathfrak{g}_P)$. Since we assumed G was reductive at the start, we know $\mathfrak{g}_P \cong \mathfrak{g}_P^\vee$ via the Killing form and this implies $\deg(\mathfrak{g}_P) = 0$. By the Riemann-Roch Theorem

$$\chi(\mathfrak{g}_P) = \deg(\mathfrak{g}_P) + \mathrm{rank}(\mathfrak{g}_P)(1-g) = \dim(G)(1-g).$$

There are two steps to proving the theorem in this case. First, we describe $H^0(X, \mathfrak{g}_P)$. In the Tannakian perspective, a section is given by a G-equivariant map $\alpha : P \to \mathfrak{g}$. For a representation $\rho : G \to GL(V)$, we get

$$a \cdot \{p, v\} = \{p, d\rho(\alpha(p)) \cdot v\}$$

where $d\rho$ is the derivative of the representation map at the identity and $\{p, v\} \in V \times_G P$. In this way,w e can describe a section a of $\mathfrak{g}_P$ as the data of $a_V \in \mathrm{End}(V_P)$ for each $V \in \mathrm{Rep}(G)$ s.t.

$$a_{V\times W} = a_V \otimes Id_{W_P} + \mathrm{Id}_{V_P} \otimes a_W.$$

Fact: The work from above actually gives a map of bundles

$$\phi_V : \mathfrak{g}_P \to \mathrm{End}(V_P).$$

To get a Tannakian description of $e \in H^1(X, \mathfrak{g}_P)$, we associate $e_V \in H^1(X, \mathrm{End}(V_P))$ using ϕ_V. The latter describes extensions of V_P by itself.

Altogether

$$H^1(X, \mathfrak{g}_P) = \left\{ \begin{matrix} \text{functor from Reg}(G) \text{ to equivalence classes of short exact sequences} \\ V \to e_V \text{ where } e_V : 0 \to V_P \to \widehat{V_P} \to V_P \to 0 \text{ and} \\ \text{this satisfies } e_{V\otimes W} = e_V \otimes W_P + V_P \otimes e_W. \end{matrix} \right\}$$

To prove the theorem, we need to determine that $T_P(\mathrm{Bun}_G)$ gives this same data. It is not hard to see that

$$\mathcal{P} \in T_P(\mathrm{Bun}_G)(S) = \left\{ \begin{matrix} \text{Exact functors Rep}(G) \to \mathrm{Vect}(X \times D \times S) \text{ given by } V \to V_{\widetilde{\mathcal{P}}} \\ \text{such that } \mathcal{P} \times_D \mathrm{Spec}(k) = \mathcal{P} \text{ and } V_{\widehat{\mathcal{P}}} \otimes W_{\widehat{\mathcal{P}}} = (V \otimes W)_{\widehat{\mathcal{P}}}. \end{matrix} \right\}$$

When $S = \mathrm{Spec}(k)$ and $\mathcal{P} = P$, computing the Baer sum, the condition on $e_{V\otimes W}$ translates into the condition on $V_{\widehat{\mathcal{P}}} \otimes W_{\widehat{\mathcal{P}}}$. Therefore, $P \in T_P(\mathrm{Bun}_G)(\mathrm{Spec}(k))$ corresponds to some $e \in H^1(X, \mathfrak{g}_P)$.

If one keeps track of the automorphisms of each extension class in $H^1(X, \mathfrak{g}_P)$ and the automorphisms of $T_P(\mathrm{Bun}_G)$, one can check that the automorphisms of P are precisely $H^0(X, \mathfrak{g}_P)$.