3/7/2025

Hairuo

Cech Cohomology

Let $\mathfrak{X}$ be a covering of X / presheaf modules Λ ring obj in T

$$\mathfrak{X} = \{X_i \to X\}_{i \in I} \qquad F \in PMod_\Lambda$$

(ring object Λ. has $+: \Lambda \times \Lambda \to \Lambda$, $m: \Lambda \times \Lambda \to \Lambda$ and some axioms)

For an ordered ~~pair~~ tuple $i = (i_0, i_1, \dots, i_r) \in I^{r+1}$

Define $X_i := X_{i_0} \times_X X_{i_1} \times_X \cdots \times_X X_{i_r}$ ("intersection")

(Cech complex) Define $C^{\cdot}(\mathfrak{X}, F) := \left(\prod_{i \in I^{r+1}}^{\deg r} F(X_i) \xrightarrow{d_r} C^{r+1}(\mathfrak{X}, F) \right)$

$$\prod_{i \in I^{r+1}} F(X_i) = C^r(\mathfrak{X}, F)$$

$$(d_r f)_{(i_0, \dots, i_{r+1})} = \sum_{j=0}^{r+1} (-1)^j f_{i_0, \dots, \hat{i_j}, \dots, i_{r+1}}$$

$$\check{H}^i(\mathfrak{X}, F) := H^i(C^{\cdot}(\mathfrak{X}, F))$$

$$\check{H}^i(\mathfrak{X}, -): PMod_\Lambda \to Ab$$

<u>Prop:</u> $\check{H}^i(\mathfrak{X}, -)$ is the <u>derived functor</u> for $\check{H}^0(\mathfrak{X}, -)$

(need abelian categories + enough injectives)

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

(let this be inj)

want LES

$$0 \to \check{H}^0(M_1) \to \check{H}^0(M_2) \to \check{H}^0(M_3)$$

$$\to R^1\check{H}^0(M_1) \to 0 \qquad \simeq \check{H}^1(M_1)$$

~~[illegible]~~ skip proof of prop.

take away, if have $\check{H}^i(I)=0$ for I inj, can do dimension shift and get an isomorphism.

$\check{H}^\bullet(\mathcal{X},F)$ is representable functor

$\check{H}^\bullet(\mathcal{X},-): PMod_\Lambda \longrightarrow Ab$

chain complex ?

$\exists\, Z_\bullet \in (PMod_\Lambda)$ s.t. $\check{H}^i(\mathcal{X},F) \cong H^i(Hom_{PMod_\Lambda}(Z_\bullet,F))$

$\check{H}^i(\mathcal{X},I)=0 \longleftrightarrow Hom(-,I)$ is exact

$$\begin{tikzcd} Mod_\Lambda \arrow[r, "RF"] \arrow[rr, bend right, "R\Gamma"'] & PMod_\Lambda \arrow[r, "R\check{H}^0"] & Ab \end{tikzcd}$$

Λ

T

(by spectral sequence $H^s(\mathcal{C}/X, \check{H}^t(F)) \Longrightarrow \check{H}^{s+t}(F)$

localized site of X

Kevin

Fibred Categories

Def: (F, p) is a fibred cat. (over C)

- $p: F \longrightarrow C$ functor between cats,
- $\exists!$

$$\begin{tikzcd} u \arrow[r, dashed, "\phi"] \arrow[d] & v \arrow[d, "p"] & \text{in } F \quad p(v) = V \\ U \arrow[r, "f"'] & V & \text{in } C \end{tikzcd}$$

- ϕ must be cartesian

$$\begin{tikzcd} w \arrow[r, dashed, "\exists!"] \arrow[rr, bend left] \arrow[d] & u \arrow[r] \arrow[d] & v \arrow[d] \\ W \arrow[r] & U \arrow[r] & V \end{tikzcd}$$

Def: $F(U)$ for $U \in C$

$F(U) =$ Ob $= u \in F$, $p(u) = U$
Mor $= u \to u'$
$p(u \to u') = id_U$

Expl: All cats are small, $Ob(-) \in Sets$, $Hom(-) \in Sets$

$Ob: Cats \longrightarrow Sets$ is a fibred cat, whose cartesian morphisms are fully faithful functors

Expl: Recall $X \times_Z Y$ it is only ! up to ! iso.

$$\begin{tikzcd} & X \times_Z Y \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

$\mathcal{P} \longrightarrow Y\text{-Sch}$, given $f: X \to Y$

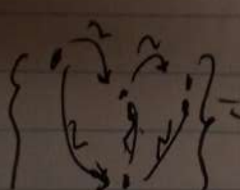
$\mathcal{P} :=$ Ob: $\underbrace{(t: T \to Y,\ P \in Y\text{-Sch},\ P \to X,\ P \to T)}_{\text{s.t.}}$

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & Y \end{tikzcd}$$

Hom: (you fill in)

$(T, P \in Y\text{-Sch})$

$\mathcal{P} \xrightarrow{p} Y\text{-Sch}$ sends to $t: T \to Y$



$= \mathcal{P}(T \to Y) = \left\{ \begin{array}{l} \text{all possible} \\ \text{fibre products } X \times_Y T \end{array} \right\}$

"Keep track of automorphisms"

Exp: (Localized ~~fibre~~ Cat)

- $\mathcal{C}$ cat, $X \in \mathcal{C}$

- $\mathcal{C}/X = \{$ Ob: $Y \to X$

  Homs: $\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$

- $\mathcal{C}/X \xrightarrow{p} \mathcal{C}$ fibred cat

  $(Y \to X) \longmapsto Y$

Thm (2-Yoneda lemma)

Given $p: F \to \mathcal{C}$ f.c., $X \in Ob(\mathcal{C})$; build $\mathcal{C}/X$

$\xi: \underbrace{Hom_{\mathcal{C}}(\mathcal{C}/X, F)}_{\text{functors of f.c.'s}} \longrightarrow F(X)$

$\in$ "Cat"
$\in$ "Sets"
$\in$ "groupoids"

$\xi(g: \mathcal{C}/X \to F) = g(X \xrightarrow{id} X)$

This is an EOC's

Notation: F = f.c.'s, write $X \to F$ as a choice of object in $F(X)$

i.e. $\mathrm{spec}\, \mathbb{C} \to X$
$\updownarrow$
a $\mathbb{C}$-pt. of X

Def: $p: \mathcal{F} \to \mathcal{C}$ a category fibred in groupoids if $(\mathcal{F}, p)$ is a f.c and

$\mathcal{F}(X)$ is a groupoid $\forall X \in Ob(\mathcal{C})$

Expl: C - curve, $r \geq 0$, $d \in \mathbb{Z}$

$Bun_{r,d}(C) \longrightarrow Sch$

$\hookrightarrow Ob = (E, S)$ where $- E$ is a v.b. on $C \times S$

$- E_s$ are v.b. of rank r degree d on C

(S: scheme)

Mor (you fill in)

$\hookrightarrow (E, S) \to (E', S')$

$- S \xrightarrow{f} S'$

$- (id \times f)^* E' \xrightarrow{\sim} E$

E.g. $\mathcal{F}(\operatorname{Spec} \mathbb{C}) = \left\{ \begin{array}{l} \text{v.b.s} \\ \text{on } C/\operatorname{Spec} \mathbb{C} \end{array} \right.$

($\mathcal{F} = Bun_{r,d}$)

/ fibred categories

Def: ~~F~~ Given two f.c.'s

$$\begin{tikzcd} F \arrow[r, "P_F"] \arrow[d, dashed, "g"'] \arrow[dr, phantom, "G"] & C \arrow[d, equal] \\ G \arrow[r, "P_G"'] & C \end{tikzcd}$$

$P_G(g(-)) = P_F(-)$

an EOFC's (equiv. of fibred cats) is a $\overbrace{\text{functor } g: F \to G}$ s.t. $\exists h: G \to F$ s.t.

$h \circ g \simeq id_F \quad g \circ h \simeq id_G$

base preserving

Fact: $g: F \to G$ functor of f.c. (over C)

Then g is EOFC's

iff $g(U): F(U) \to G(U)$ are EOC's $\forall U \in C$

Reminder for Sch

Sch $\longrightarrow$ Functors Sch $\to$ Sets

$X \longrightarrow h_X(-) = Hom_{Sch}(-, X)$

Yoneda Embedding: Sch $\hookrightarrow$ ~~Nat. Trans. of~~ Functors Sch $\to$ Sets

$Hom_{Sch}(X, Y) \to Hom(h_X, h_Y)$

To understand X, study h_X

It also enlightens you to what is the right def of things!

Cat. fibred in Sets

<u>Thm:</u>

$p: F \longrightarrow C$ fibred cat, $F(X)$ is a set $\forall X \in C$

Grothendieck const. $\leftarrow$ $\Gamma: \begin{pmatrix} \text{presheaves} \\ \text{on } C \end{pmatrix} \longrightarrow \begin{pmatrix} \text{category} \\ \text{fibred in sets} \end{pmatrix}$

$C^{op} \longrightarrow Sets \quad \longmapsto \quad p: F \longrightarrow C$

$\mathcal{F}$

$ob(F) = (U, u \in \mathcal{F}(U))$

$p((U,u)) = U$

Γ is an EOC

$+$DD (descent data)

gluing cond. for sections,

get a sheaf.