1. Grothendieck Toplogies and Sites - Hairuo

Throughout, C denotes a fixed category.

Definition 1.1. A **Grothendieck topology** on C is a collection $\mathrm{Cov}(X)$ for every object X of C consisting of collection of morphisms $\{X_i \to X\}$ such that

(1) if $\varphi : Y \to X$ is an isomorphism, then $\{Y \to X\}$ is in $\mathrm{Cov}(X)$,
(2) if $\{X_i \to X\} \in \mathrm{Cov}(X)$, and $Y \to X$ is amorphism, then the fibre product $X_i \times_X Y$ exists and $\{X_i \times_X Y \to Y\}$ is in $\mathrm{Cov}(Y)$,
(3) if $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$, and for all i, there is a covering $\{V_{ij} \to X_i\} \in \mathrm{Cov}(X_i)$, then $\{V_{ij} \to X_i \to X\}_{i,j} \in \mathrm{Cov}(X)$.

A **site** is a category C together with a Grothendieck topology.

Example 1.2. Let C be the category of open subschemes of a fixed scheme X with inclusions being the morphisms of this category.

Then $\{U_i \to U\} \in \mathrm{Cov}(U)$ iff $\bigcup_{i\in I} U_i = U$ iff $\coprod_{i\in I} U_i \to U$ is surjective.

This site is denoted X_{zar} is often referred to as a the **small Zariski site**.

Example 1.3. Let $C := \mathrm{Sch}$. Then $\{U_i \to X\}$ is a covering iff $U_i \to X$ is an open immersion for all i and $\coprod_{i\in I} U_i \to X$ is surjective. This is known as the **big Zariski site**.

Example 1.4. Let C be a site, and $X \in C$ an object. The localized site C/X has underling categorhy consisting of X-objects $Y \to X$ and morphisms being X-morphisms. The underling Grothendieck topology consists of sets of X-morphisms $\{U_i \to U\}$ such that $\{U_i \to U\}$ is a covering in C. This is called the **localized site** (at X).

Example 1.5. Let $\mathrm{Et}(X)$ denote the full subcategory of X-schemes consisting of objects $U \to X$ such that $U \to X$ is a étale morphisms. The fact that this is a full subcategory is a consequence of Olsson Proposition 1.4.3(iii). Coverings of an object $U \to X$ consist of a collection of morphisms $\{U_i \to U\}$ s.t. $\coprod_{i\in I} U_i \to U$ is surjective. This is called the **small étale site**.

Example 1.6. The **big étale site** is define analogous to how the big Zariski site is defined. The difference here is the category simply consists of all scheme, the morphisms are not required to be étale, but the coverings consists of $\{U_i \to U\}$ s.t. $U_i \to U$ is étale and $\coprod_{i\in I} U_i \to U$. Typically, say that the cover is surjective if this latter property holds.

Example 1.7. The **fppf site** and **fpqc site** are defined in a similar sense except the covers of the former to have morphisms which are faithfully flat and locally of finite presentation while the latter has morphisms which are faithfully flat and quasicompact.

Definition 1.8. A **presheaf** on a category C is a functor $C^{op} \to \mathrm{Set}$. Let $\widehat{C}$ denote the category of presheaves on C.

A **separated presheaf** on a site C is a presheaf on C s.t. for all coverings $\{U_i \to U\}$, the obvious map of sets $F(U) \to \prod_{i\in I} F(U_i)$ is an inclusion.

A **sheaf** on a site C is a presheaf such that

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j).$$

is exact.

The reader should be aware that the natural forgetful functors Sheaves $\to$ SepPresheaves $\to$ Presheaves compose and it has a left adjoint which is called sheafification. In fact, the two forgetful functors have a left adjoint and the proof essentially follows the usual proof that sheafification functors exist (via the +-construction).

Definition 1.9. A **topos** is a category that is equivalent to a category of presheaves of sets on some site.

Let $\mathrm{Et}^{aff}(X)$ denote the full subcategory of $\mathrm{Et}(X)$ whose objects are all X-morphisms $U \to X$ with U affine, and make it a site by declaring a collection of morphisms $\{U_i \to U\}$ to be a covering iff it is a covering in $\mathrm{Et}(X)$.

Theorem 1.10. The associated topos on $\mathrm{Et}^{aff}(X)$ is equivalent to the topos on $\mathrm{Et}(X)$.

There is a notion of a continuous map of sites – it is a functor which sends coverings to coverings and commutes with fibre products whenever they exist.

Definition 1.11 (Global sections functor)**.** Suppose X_{Et} is the topos on the small étale site of a scheme X. Let Λ be a ringed object in X_{Et}. Let Mod_Λ be the category of Λ-module objects in X_{Et}.

Theorem 1.12. Mod_Λ is an abelian category with enough injectives.

Olsson's book proves this theorem under the assumption that the topos "has enough points" so that they can mimick Hartshorne's proof.

An alternative construction of the global sections functor can be made using localized sites. Let T be the topos. Then define the global sections functor (relative to $X \in C$) by

$$\Gamma : F \mapsto \mathrm{Hom}_T(h_X, F) =: \Gamma(T, F).$$