Tangent-Obstruction Theories and Smoothness

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Motivation: Infinitesimal Lifting

Theorem (Infinitesimal Lifting Property)

Let $Spec(A)$ be smooth of finite type over k,
$f : Spec(B) \to Spec(A)$ be a morphism of k-schemes, and Y' be a scheme such that $Spec(B)$ is a closed subscheme defined by a nilpotent ideal sheaf. Then there exists a morphism $f' : Y' \to Spec(A)$ such that $f'|_{Spec(B)} = f$.

Remarks

1. Exercise: Y' above is actually affine.
2. To rephrase: If $Spec(A)$ is a smooth affine scheme and $B' \twoheadrightarrow B$ is a "nilpotent thickening" as above, then the natural map $Hom(A, B') \to Hom(A, B)$ is surjective. All that follows explores this interpretation.

Proof of Infinitesimal Lifting: Preliminaries

Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & & & & \overset{\exists ?}{\nwarrow} & \uparrow {\scriptstyle f} & & \\ & & & & & & A & & \end{array}$$

Lemma (Exercise). If any lift θ of f exists, and $\delta : A \to I$ is a k-derivation, then $\theta + \delta$ is another such lift. Conversely, if $\theta_{1,2}$ are two lifts, then $\theta_1 - \theta_2$ is a k-derivation.

Thus the set of possible lifts is either empty or has a simply transitive action by $Hom_A(\Omega_{A/k}, I)$ (independent of smoothness).

Claim (Exercise). It is enough to show the theorem in the case $I^2 = 0$.

Proof of Infinitesimal Lifting: Main Construction

Now there exists a polynomial ring P with a surjection $P \twoheadrightarrow A$ and a map $h : P \to B'$. These fit into a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle f} & & \\ 0 & \longrightarrow & J & \longrightarrow & P & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Thus $h|_J$ sends J to I; as $I^2 = 0$, $h|_J$ factors through J/J^2. Call this map (of P-modules) $\overline{h}$. Both of these also have a compatible A-module structure.

Proof of Infinitesimal Lifting: Smoothness

Now: J/J^2 is the *conormal sheaf* of $Spec(A)$ in $Spec(P)$. As A is smooth, we have an exact sequence of A-modules (in general, it is only right-exact):

$$0 \longrightarrow J/J^2 \longrightarrow \Omega_{P/k} \otimes_P A \longrightarrow \Omega_{A/k} \longrightarrow 0$$

Taking the associated long exact sequence to $Hom_A(-, I)$, we see that $Hom(\Omega_{P/k} \otimes A, I) \to Hom(J/J^2, I)$ is surjective if $Ext^1(\Omega_{A/k}, I) = 0$. But this is the case because A is smooth, so every map $g : J/J^2 \to I$ lifts to $Hom(\Omega_{P/k} \otimes A, I)$. As the A and P-module structures on I agree, $Hom(\Omega_{P/k} \otimes A, I) = Hom_P(\Omega_{P/k}, I)$.

Proof of Infinitesimal Lifting: Conclusion and Remarks

In particular, $\overline{h} : J/J^2 \to I$ lifts to a k-derivation $\theta : P \to I$. Recall $\overline{h}$ came from a ring homomorphism $h : P \to B'$. The conclusion follows from

Claim (Exercise). $h - \theta : P \to B$ is a ring homomorphism that descends to A and lifts f.

Remarks.

1. We see the action of $Hom_A(\Omega_{A/k}, I)$ in the choice of lift for $\overline{h}$. By the first lemma, the resulting lifts are independent of P.

2. We want to extend this to deformation functors: how does a deformation functor behave under nilpotent thickenings? This is partially answered by *tangent-obstruction theories*: the "obstruction" detects when a lift is possible and the "tangent" parametrizes lifts. The idea is that a functor "is" smooth when lifting is possible, i.e. the obstruction vanishes.

Tangent-Obstruction Theories

Let $D : Art_k^{loc} \to \mathbf{Set}$ be a deformation functor.

A *small extension* in Art_k^{loc} is a surjection $p : B \to A$ with $ker(p) := M$ one-dimensional over k.

Definition

A *tangent-obstruction theory* for D is a pair (T^1, T^2) of f.d. k-vector spaces satisfying:

- For any small extension, there is a functorial exact sequence

$$T^1 \otimes M \longrightarrow D(B) \longrightarrow D(A) \xrightarrow{ob} T^2 \otimes M$$

- $ob(a) = 0$ iff a lifts to $D(B)$.
- If a lifts, there is a transitive action of $T^1 \otimes M$ on the set of lifts.
- If $A = k$, the action is simply transitive.

Example 1: Complete Local k-algebra

Proposition.

Let S be a complete Noetherian local k-algebra such that $dim_k(\mathfrak{m}_S/\mathfrak{m}_S^2) = d$ and let $p : k[[x_1, ..., x_d]] \to S$ be a surjection (exercise: this always exists) with kernel J. Let $T^1 := Hom_S(\mathfrak{m}_S/\mathfrak{m}_S^2, k)$, $T^2 := Hom_S(J/\mathfrak{m}_S J, k)$. Then (T^1, T^2) is a tangent-obstruction theory for $h_S := Hom(S, -)$.

Proof in the notes (discussion after Definition 2.7).

Remarks.

1. Recall that any such an S is smooth iff it is isomorphic to $k[[x_1, ..., x_d]]$ – exactly when T^2 vanishes. Hence “obstruction”.

2. Notice that T^1 is the tangent space of S. This is not a coincidence; it follows from considering $A = k$ (i.e., $T^1 = D(k[\epsilon]/(\epsilon^2))$). Hence “tangent”.

Example 2: Smooth Proper Schemes

Proposition.

Let X be a smooth proper scheme. Let D be the functor that sends $A \in Art_k^{loc}$ to the set of X_A, flat over A, with fixed isomorphism $\varphi_A : X_A \times_A k \simeq X$. Then $T^i := H^i(X, T_X)$ is a tangent-obstruction theory for D.

We first study the deformations of affine schemes X (i.e. instead of smooth proper X, consider smooth affine X). The idea is to patch these together to deduce the general case.

Lemma.

Let D be as above with X affine. Then $X' \in D(A)$ is the *trivial deformation* $X \times Spec(A)$.

Example 2: Smooth Proper Schemes. Proof of Lemma

By the infinitesimal lifting property, the isomorphism $\varphi : Y' \times_A k \simeq Y$ lifts to a morphism $\varphi' : Y' \to Y$. Note that inverting φ and composing with the projection $Y' \times A \to Y'$ gives a morphism $i : Y \to Y'$ such that $\varphi' \circ i = id$. Translating to ring theory terms, is φ' gives a splitting for the extension $R' \twoheadrightarrow R$.

Example 2: Smooth Proper Schemes, T^1

We now identify T^1 and T^2 for D. Cover X (smooth, proper) by affines U_i. We are interested in schemes X' over $k(\epsilon)/(\epsilon^2)$ together with a fixed isomorphism $\varphi : X' \times_{k(\epsilon)/(\epsilon^2)} k \simeq X$. We want to glue from affines U_i' over $k(\epsilon)/(\epsilon^2)$. By the lemma, we have $f_i : U_i' \simeq U_i \times k(\epsilon)/(\epsilon^2)$, where we can change trivialization by $Aut_{U_i}(U_i') = \Gamma(U_i, U_i \times (\epsilon)) = H^0(U_i, T_{U_i})$. On overlaps we have transition functions g_{ij}. They are compatible exactly when they satisfy a cocycle condition. If they do, this defines a Čech 1-cocycle. Hence, $T^1 = H^1(X, T_X)$.

Example 2: Smooth Proper Schemes, T^2

We see that no global lift exists if the g_{ij} do not satisfy a cocycle condition; it can be off by a 2-cocycle. Thus $H^2(X, T_X)$ is a candidate for T^2. Observe that the above argument goes through over any base $A \in Art_k^{loc}$ with $B \to A$ a small extension with the difference that $Aut_{U_i}(U_i') = H^0(U_i, U_i \times M)$. Globalizing and applying the projection formula (the $M \times U_i$ glue to a vector bundle over X) identifies the obstruction class as lying in $H^2(X, T_X) \otimes M$ (details are in the notes, Theorem 3.4).

Example 3: Coherent Sheaves (Vector Bundles)

Proposition.

Let X be a smooth proper scheme and $\mathcal{E}$ be a coherent sheaf (vector bundle) on X. Let D be the functor that sends $A \in Art_k^{loc}$ to the set of coherent sheaves (vector bundles) $\mathcal{E}_A$ on $X \times A$, flat over A, with a fixed isomorphism $\varphi_A : \mathcal{E}_A \otimes_A k \simeq \mathcal{E}$. Then $T^i := Ext^i(\mathcal{E}, \mathcal{E})$, $i = 1, 2$ is a tangent-obstruction theory for D.

Dima proved this last time for vector bundles. The argument is similarly by patching trivial deformations. Dima also proved that $Ext^1(\mathcal{E}, \mathcal{E})$ was the tangent space of D.

In general, $Ext^2(\mathcal{E}, \mathcal{E})$ being the obstruction space relies on deeper homological techniques because we can't appeal to local triviality.

Application to Moduli Spaces

Corollary.

Let C be a smooth projective curve of genus g. The moduli space of smooth curves of genus g, $\mathcal{M}_g$ ($g \geq 2$), is smooth at $[C]$. The tangent space to $\mathcal{M}_g$ at a curve $[C]$ is identified with $H^1(C, T_C) = H^0(C, (\Omega^1_C)^{\otimes 2})^*$ (Serre duality), which has dimension $(4g-4)+1-g = 3g-3$ (Riemann-Roch).

Corollary.

Let X be a smooth projective curve of genus $g \geq 2$. For a stable rank-n vector bundle $\mathcal{E}$, the moduli space of stable rank-n vector bundles on X is smooth at $[\mathcal{E}]$ and has tangent space $H^1(X, \mathcal{E}nd(\mathcal{E})) = H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes K_X)^*$ at $\mathcal{E}$, which has dimension $n^2(g-1)+1$ (Riemann-Roch for vector bundles).

Pro-representability, Formal Smoothness, and Hulls

Definition.

A morphism of deformation functors $\alpha : F \to G$ is (formally) *smooth* if for $B \to A$ a small extension we have $F(B) \to F(A) \times_{G(A)} G(B)$ is surjective.

Definition.

A deformation functor D is *pro-representable* if there exists a complete Noetherian local k-algebra R and an isomorphism $D \simeq Hom(R, -)$.

Definition.

A *hull* for D is a complete Noetherian local k-algebra R and a smooth morphism $\alpha : Hom(R, -) \to D$ which is bijective on $k(\epsilon)/(\epsilon)^2$.

Baby Schlessinger

Theorem.

A hull exists for D iff D has a tangent-obstruction theory $T_D^{1,2}$, and furthermore, D is pro-representable iff

$$0 \to T_D^1 \otimes M \to D(B) \to D(A) \to T_D^2 \otimes M$$

is left-exact for all small extensions $B \to A$.

Proof (sketch). For the second part, we need to show that the surjection $h_S(B) \to h_S(A) \times_{D(A)} D(B)$ is a bijection. But if the tangent-obstruction sequence is left-exact, $T_D^1 \otimes M$ acts simply transitively on lifts from $D(A)$ and this is identified with $T_S^1 \otimes M$ (the bijection on tangent spaces guarantees this). This in turn implies the second part.

Proof of Baby Schlessinger

$h_S := Hom(S,-)$ has $T^1_S = (\mathfrak{m}_S/\mathfrak{m}_S^2)^*$, $T^2_S = (J/\mathfrak{m}_S J)^*$ a tangent-obstruction theory. Pick S smooth with $T^1_S \simeq T^1_D$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^1_S \otimes M & \longrightarrow & h_S(B) & \longrightarrow & h_S(A) & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & T^1_D \otimes M & \longrightarrow & D(B) & \longrightarrow & D(A) & \longrightarrow & T^2_D \otimes M \end{array}$$

$T^1_S \otimes M$ acts transitively on lifts from $D(A)$ in $D(B)$ while it acts *simply transitively* on lifts from $h_S(A)$ in $h_S(B)$. This defines a surjection $h_S(B) \to D(B) \times_{D(A)} h_S(A)$. Taking $A = k$ and composing small extensions defines surjections $h_S(B) \to D(B)$ for all B, hence a morphism. Since we have this for all small extensions, such an S defines a hull for D.

Looking Ahead

- We will first be interested in representability and smoothness properties of deformation functors. The first major result is Schlessinger's criteria for pro-representability/hulls (which is closely related to the "Baby Schlessinger" result above).
- If we have some pro-representability result, we are interested in extending it to something global. There is a nice example of this in the notes with an application to Picard groups (Theorem 2.14). Artin's paper "Algebraization of Formal Moduli I" provides some very general theory.

Thank you!