Fifth Annual UW Madison undergraduate math competition

April 17th 2019, VVB135, 5:30PM - 8:00PM

1. Determine all 2×2 real matrices X (i.e. $X \in M_2(\mathbb{R})$) that satisfy the following equality
$$X^{2019} + 2X = \begin{pmatrix} 3 & 0 \\ 2012 & 3 \end{pmatrix}.$$

2. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function satisfying
$$\int_0^1 f(x)\,dx = \frac{2^{2018} - 1}{2018}.$$
Prove there exists $a \in [0, 1]$ such that $f(a) = (a + 1)^{2017}$.

3. We call a positive integer number N *nice* if N has an even number of digits, and has the property that if we insert one multiplication sign in the exact middle of its digits, then we get that the result of the multiplication is a divisor of N. Identify all nice numbers. (For example $N = 12$ is nice since $1 \cdot 2 = 2$ divides 12. The number 1221 is not nice, because $12 \cdot 21 = 252$ does not divide 1221.)

4. A and B play the following game. They roll a fair die repeatedly and write down the outcomes. They stop if there is a k so that the sum of the last k outcomes is equal to 2018 or 2019. A wins if there is a k for which the sum is 2018, B wins if there is a k for which the sum is 2019, it is a tie if there are sums for both 2018 and 2019.

   a) Show that with probability one the game will end at some point.
   b) Who has a larger probability to win?

5. For a positive integer n let a_n denote the number of ways we can cover a $3 \times (2n)$ board with 2×1 dominos. (E.g. $a_1 = 3$.) Show that
$$a_n \geq \frac{7^n}{2^{n+1}}.$$

6. Let $f(x, y)$ be a real polynomial of degree 5, that is,
$$(x, y) = \sum_{\substack{0 \leq i,j \\ i+j \leq 5}} c_{ij} x^i y^i,$$
where c_{ij} are constant real numbers. What is the largest number of zeros that f can have among all non-negative integer pairs (x, y) with $x + y <= 100$?

7. Consider a thin and infinitely stretchable rubber band held taut along an x-axis with its left endpoint fixed at $x = 0$ and its right endpoint initially at $x = c$, $c > 0$. At time $t = 0$ the band starts to stretch uniformly and smoothly in such a way that the left endpoint remains stationary at $x = 0$ while the right endpoint moves away from the left endpoint with constant speed $v > 0$. A small ant leaves the left endpoint at time $t = 0$ and walks steadily and smoothly along the band towards the target point at a constant speed $\alpha > 0$ relative to the point on the band where the ant is at each moment. Will the ant ever reach the right endpoint?