SIXTH ANNUAL UW MADISON UNDERGRADUATE MATH COMPETITION

1. Suppose $f(x)$, $g(x)$, $h(x)$ are differentiable functions on an interval $[a, b]$. Show that there exists $c \in (a, b)$ such that

$$\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{vmatrix} = 0.$$

2. Suppose that N people sit at a round table. They get up and then sit back down again, not necessarily in their original chairs (but all chairs are occupied again). We would like to shift everybody at the table k sits clockwise so that at least two persons are in their original seats. Show that such k always exists if N is even, and conversely, if N is odd, such k may fail to exist. ($k = 0$ is allowed.)

3. Let $S = \{2^a 3^b \mid a, b \in \mathbb{Z}_{\geq 0}\}$. Find all solutions of $x + y = z$ with $x, y, z \in S$.

4. Three tennis players have a tournament according to the following scheme. In the first game, A plays with B, and from then on, the winner of the last game play with the participant who did not play in the last game. A person who wins two games in a row wins the tournament.

Assume that in each game, the two players have equal chance of winning. Find the probability of each participant's victory in the tournament.

5. A polynomial $f(x)$ of degree n satisfies

$$f(2^k) = k \qquad \text{for all } k = 0, \ldots, n.$$

Show that

$$f'(0) = 2 - \frac{1}{2^{n-1}}.$$

6. Let $f : [0, 1] \to [0, 1]$ be a continuous bijective function such that $f(0) = 0$. Denote by f^{-1} the inverse function of f.

Show that for any $\alpha \geq 0$ the following inequality holds:

$$(\alpha + 2) \cdot \int_0^1 x^\alpha \left(f(x) + f^{-1}(x)\right)\, dx \leq 2.$$

7. Let n be a positive integer. As usual, let S_n the set of all permutations of the set $\{1, \ldots, n\}$. Prove that

$$\sum_{\sigma \in S_n} \frac{1}{\sigma(1)} \cdot \frac{1}{\sigma(1) + \sigma(2)} \cdots \frac{1}{\sigma(1) + \sigma(2) + \cdots + \sigma(n)} = \frac{1}{n!}.$$

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