SEVENTH ANNUAL UW MADISON UNDERGRADUATE MATH COMPETITION

1. Let f be a bijection of a finite set X. Prove that the number of sets $A \subseteq X$ such that $f(A) = A$ is the integer power of 2.

2. Let

$$I(x) = \int_0^{\pi/2} \sin^x t dt, \quad \text{for} \quad x > 0.$$

Prove that for $x > 1$, the function $f(x) = xI(x)I(x-1)$ is periodic.

3. Prove that there do not exists integers m, n, r such that $n(n+1)(n+2) = m^r$ and $n \geq 1, r \geq 2$.

4. We are given a system of 10 linear equations in 50 variables $x_1, \ldots, x_{50}$:

$$a_{1,1}x_1 + a1, 2x_2 + a_{1,3}x_3 + \cdots + a_{1,50}x_{50} = 0,$$

$$\cdots$$

$$a_{10,1}x_1 + a_{10,2}x_2 + a_{10,3}x_3 + \cdots + a_{10,50}x_{50} = 0.$$

Here all coefficients $a_{i,j}$ are two-digit numbers (in particular, positive integers). Show that there exists an integer solution $x_1, \ldots, x_{50}$ that is non-trivial (that is, not all x_i are zero) and such that $|x_i| < 10$.

5. Suppose that the continuous on $[0, 1]$ function f is positive at interior points and vanishes at the ends. Show that there exists a square whose two vertices lie on the x-axis, and the other two on the graph $y = f(x)$.

6. Let A be an $n \times n$ matrix, with entries chosen independently at random. The random law used to choose each entry of A may differ between entries. Show that if, for each column of A, the expected value of the sum of its entries is 0, then the expected value of $\det(A)$ is 0.

Date: April 15, 2023.