EIGHTH ANNUAL UW MADISON
UNDERGRADUATE MATH COMPETITION

1. Compute

$$\int_0^{\pi/2} \sin^2(\sin(t)) + \cos^2(\cos(t))\, dt.$$

2. Find all real polynomials $p(t)$ such that

$$p(u^2 - v^2 + 1) = p(u + v + 1) \cdot p(u - v + 1)$$

for all $u, v \in \mathbb{R}$.

3. Let p be a prime number. Fermat's Little Theorem states that every invertible $a \in (\mathbb{Z}/p\mathbb{Z})$ satisfies

$$a^{p-1} = 1.$$

In fact, $p - 1$ is the minimal power that works.

(a) Show that in this form, the theorem fails for matrices, and even for 2×2-matrices: there is an invertible 2×2 matrix A with entries in $\mathbb{Z}/p\mathbb{Z}$ such that

$$A^{p-1} \neq I.$$

(b) Find the minimal positive exponent $N > 0$ such that

$$A^N = I$$

for any A as in part (a). (For partial credit, find some $N > 0$ that works.)

4. Show that π can be obtained as the limit of a sequence of numbers of the form

$$\sqrt{x^2 + y^4} - z,$$

where x, y, and z are all integers.

5. Let $P(x)$ be a non-constant polynomial with integer coefficients. Show that there exists an integer N such that the numbers

$$|P(N)|, |P(N+1)|, \ldots, |P(N+2024)|$$

are all composite. (By convention, 0 and 1 are not considered composite.)

6. Let $\mathcal{C}$ be a circle of radius 1. Let us inscribe a random triangle in $\mathcal{C}$ by choosing, randomly and independently, three points on $\mathcal{C}$. (Each point is uniformly distributed on $\mathcal{C}$.) What is the expected value of the area of the triangle?

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