graph isomorphism problems

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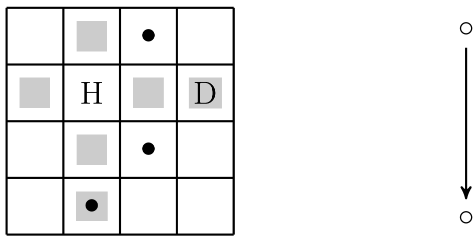
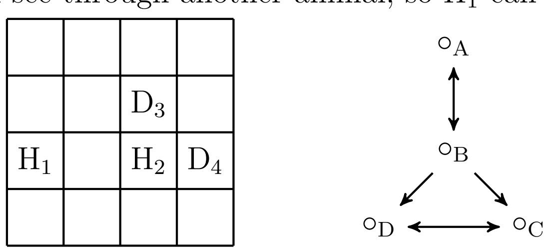
Problem 0: Heron, Dingo, Badger

On Planet Flagellan there is a large meadow where Badgers and Dingoes and Herons all live together. These animals hardly ever move, and some Flagellans even make maps showing the positions of the animals. They also have *directional vision*. For example, Herons can only see along straight lines in the horizontal and vertical directions, while Dingoes can see only along diagonals. Let's look at some maps now. The first map shows a Heron and a Dingo.



In the map, the spaces the Heron can see are shaded, while the spaces the Dingo can see have dots. To the right of the map is something called a *Sight-Graph* containing two dots and an arrow. The two dots represent the two animals in the map, and the arrow points from one animal to another that it can see. By examining the map we notice that the Heron can see the Dingo but not vice versa. This is why the arrow only goes in one direction. The Heron must correspond to the upper dot in the Sight-Graph. The locations of the dots do not matter (an "upper" dot can represent an animal on the bottom part of the map).

Next, here are two Herons and two Dingoes, labeled with numbers. No animal can see through another animal, so H_1 can't see D_4.



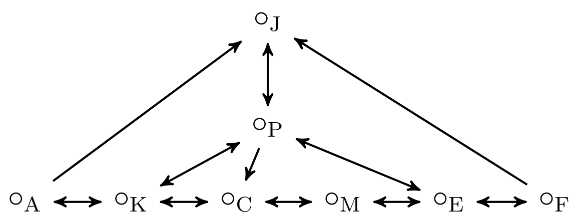
This Sight-Graph has four dots representing the four animals. Let's figure out how it relates to the map. For example, H_2 can see all of the other animals on the map, so it must go with the middle dot, B. Then since H_1 can see H_2, we know that H_1 goes with dot A. Finally, C and D must be the two Dingos, although we can't tell which is which.

Badgers have the strongest vision: they can see like Herons and also like Dingoes.

Part 1: Warmup: Maps and Graphs

Question 0.1: (2 points) Here is a map with a corresponding Sight-Graph. The animals are labeled with numbers and the dots are labeled with letters.

D_8			H_6
	B_9		H_5
H_1	H_2	H_3	H_4



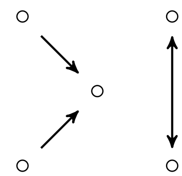
Which animal goes with dot J?

Which dot goes with the Heron H_4?

Part 2: Completions

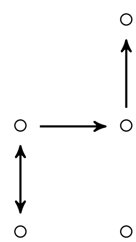
Question 0.2: (2 points) Add two Herons to the map so that the Sight-Graph is correct. *Hint: In the sight graph there is a two-way arrow. What part of the map corresponds to that?*

		D	
		D	
			D



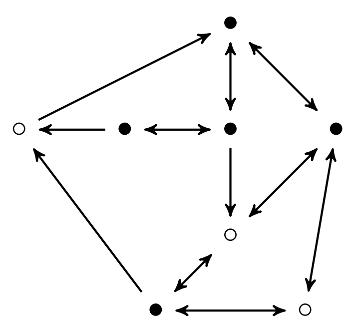
Question 0.3: (2 points) Add some arrows to complete the Sight-Graph. *Hint: On the map, only one pair of animals can see each other.*

H		H	
	D	D	
			H



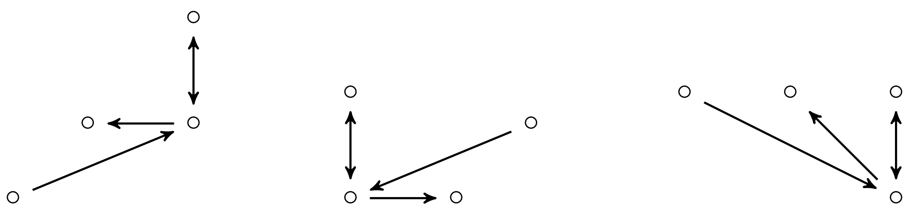
Question 0.4: (2 points) Add three animals to the map so that the Sight-Graph is correct. The five solid dots in the Sight-Graph represent the animals already on the map. *Hint: one of the five solid dots has no arrows to or from the other four solid dots.*

H			B
H			
H			
B			



Part 3: Which are the same?

Remember that the locations of dots in a Sight-Graph do not matter. This means that these three Sight-Graphs are all the same, even if they look a little different:



Another way to say this is that *two Sight-Graphs are the same if you can move the dots in one to make it look like the other one.* The arrows stay attached to the dots while they move.

Question 0.5: (3 points) Here are two maps. Are their Sight-Graphs the same? (Yes or No)

H		H	
H		B	
		D	

	D		
D	H		D
	D		

Question 0.6: (3 points) Here are two maps. Are their Sight-Graphs the same? (Yes or No)

	H			
	B			
H	H			
	H			D

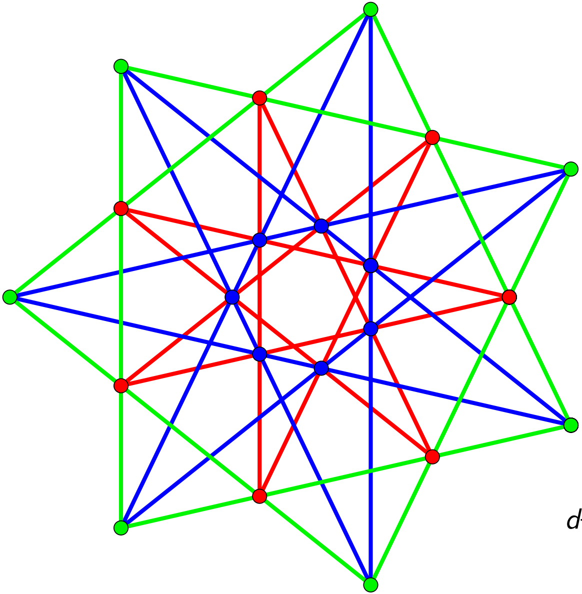
	D	D	H
H	H		B

A *geometric 4-configuration* is a collection of lines and points in the Euclidean plane such that:

- each line passes through exactly four of the points, and
- each point lies on exactly four of the lines.

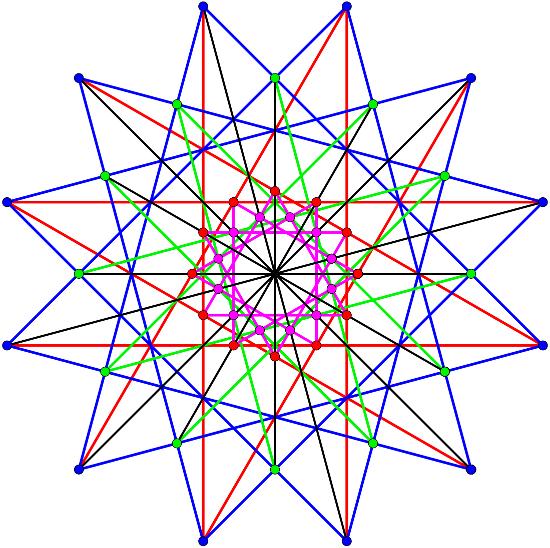
Can you think of an example?

Grünbaum, Branko, and J. F. Rigby *The Real Configuration* (21_4).
J. London Math. Soc (2) 41 (1990)



21 pts
21 lines
d_7 symmetry

Berman, Leah Wrenn and W. H. Mitchell, *Sparse Line Deletion Constructions for Symmetric 4-Configurations*, Ars Contemporanea Mathematica (9), 2 (2015)



Several infinite families of 4-cfgs are known, plus sporadic examples.

- If $n \geq 18$ and $n \neq 19, 22, 23, 26, 37, 43$,
  then there is a 4-cfg with n points and lines.
- There are none with $n \leq 17$.

Finding new examples: "more of an art than a deductive science." [Grünbaum p.169]

We'll consider *celestial* 4-configurations.

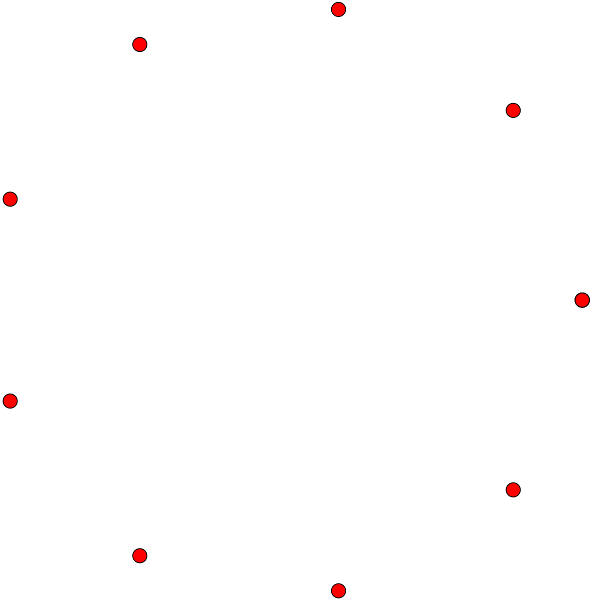
Each is described by a symbol

$$m\#(s_1, t_1; \cdots ; s_k, t_k),$$

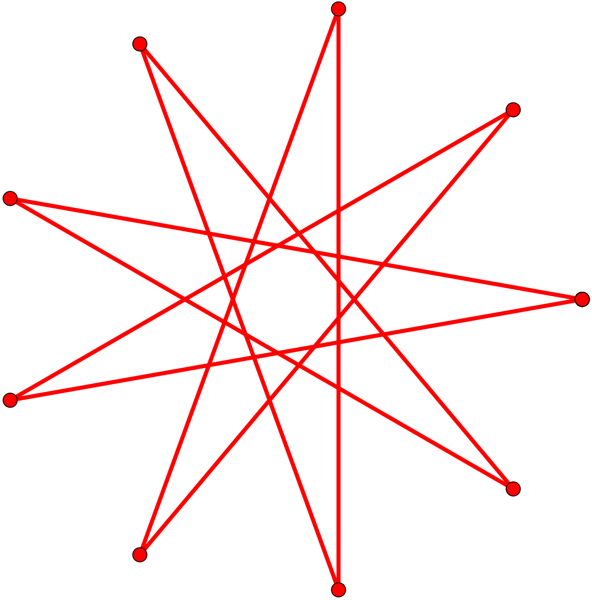
e.g.

$$9\#(4, 3; 1, 3; 2, 3).$$

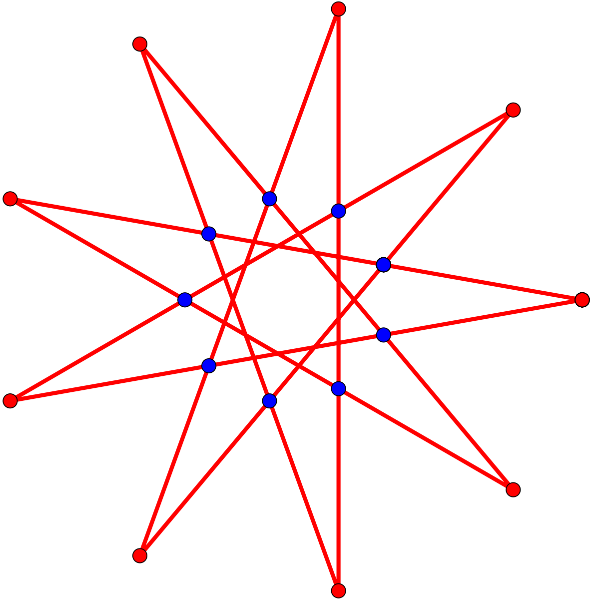
9 $\#(4, 3; 1, 3; 2, 3)$



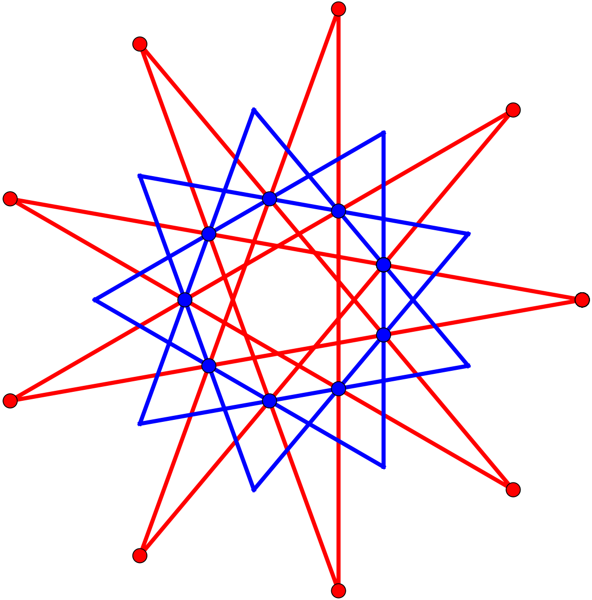
$9\#(\boxed{\mathbf{4}}, 3; 1, 3; 2, 3)$



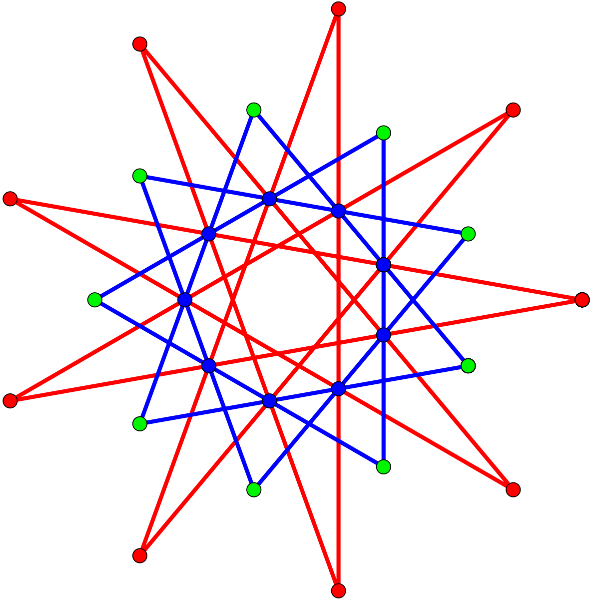
$9\#(4, \boxed{\mathbf{3}}; 1, 3; 2, 3)$



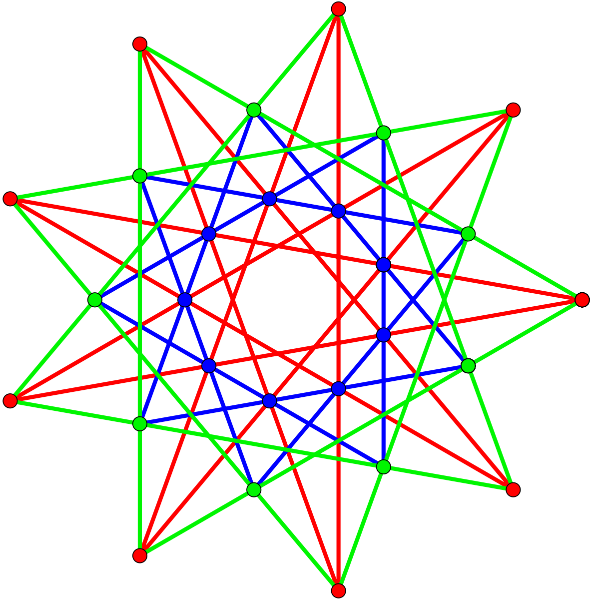
$9\#(4, 3; \boxed{\mathbf{1}}, 3; 2, 3)$



$9\#(4,3;1,\boxed{\mathbf{3}};2,3)$



$9\#(4,3;1,3;\boxed{\mathbf{2}},3)$

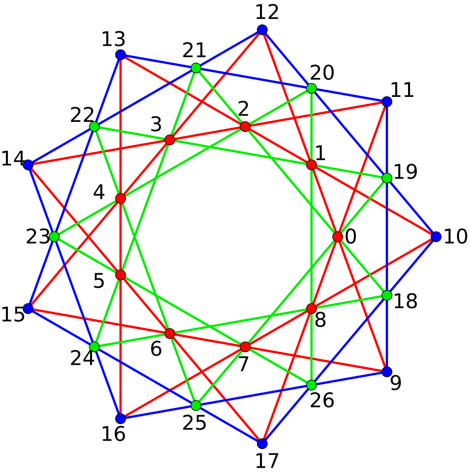


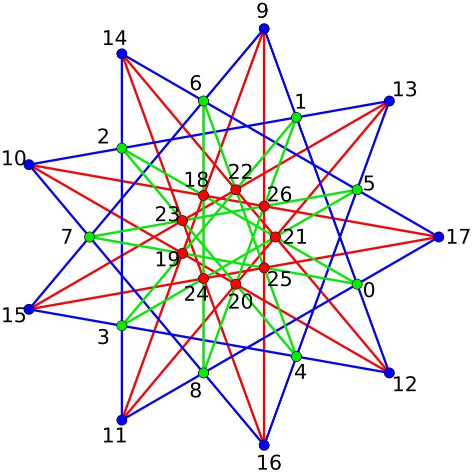
Does this construction always give a cfg?
No.
It works iff $m\#(s_1, t_1; s_2, t_2; \cdots ; s_k, t_k)$ satisfies

1. $m \geq 7, k \geq 2$
2. $s_j, t_j < \frac{m}{2}$
3. $\sum_j s_j + t_j$ is even
4. $s_1 \neq t_1 \neq s_2 \neq \cdots t_k \neq s_1$
5. $\prod_j \cos\left(\frac{s_j \pi}{m}\right) = \prod_j \cos\left(\frac{t_j \pi}{m}\right)$
6. substring condition

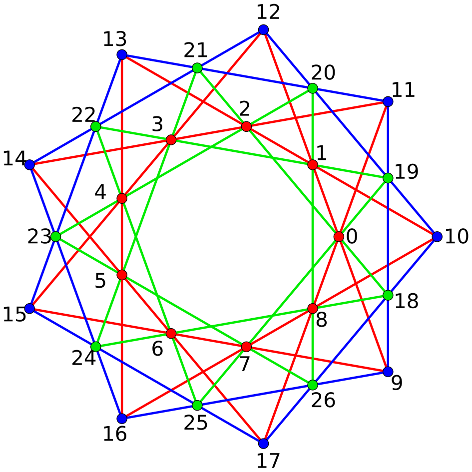
Problem: Which celestial cfgs are isomorphic?

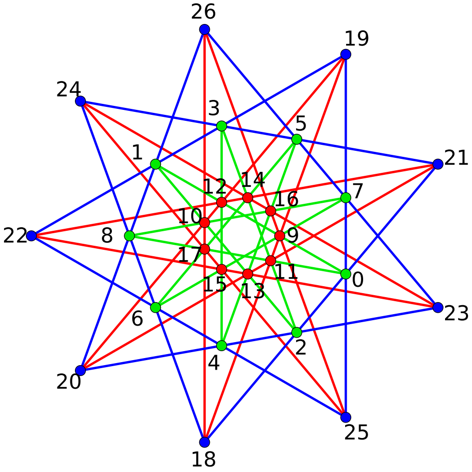
"isomorphic" $=$ graph-theoretical notion (ignore geometry)



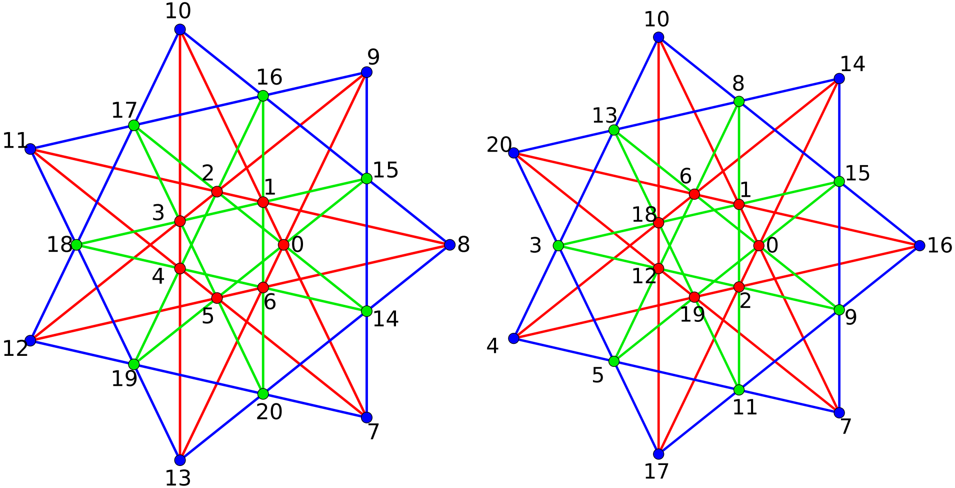


$9\#(1,3;2,1;3,2) \qquad \cong \qquad 9\#(2,4;3,2,4,3)$





$9\#(1,3;2,1;3,2) \quad \cong \quad 9\#(1,4;3,1;4,3)$



an automorphism of $7\#(1,3;2,1;3,2)$
fixing 0,1,7,10,15

Partition these:

$9\#(3,2;1,3;2,1)$

$9\#(4,3;2,4;3,2)$

$9\#(4,3;1,4;3,1)$

$9\#(4,2;1,4;2,1)$

$9\#(4,3;2,3;1,3)$

$9\#(4,3;1,3;2,3)$

Solution:

$9\#(3,2;1,3;2,1)$

$9\#(4,3;2,4;3,2)$

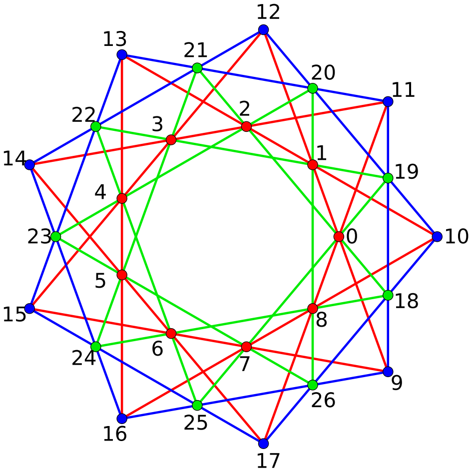
$9\#(4,3;1,4;3,1)$

$9\#(4,2;1,4;2,1)$

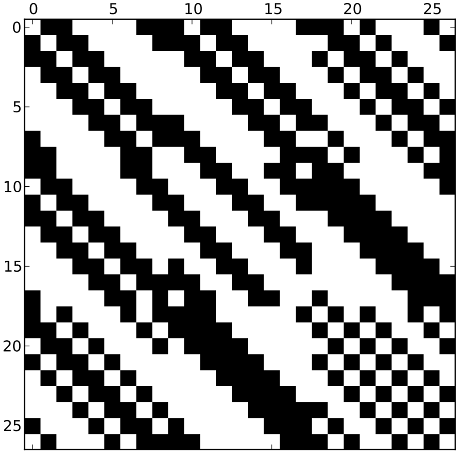
$9\#(4,3;2,3;1,3)$

$9\#(4,3;1,3;2,3)$

...whence negative results?

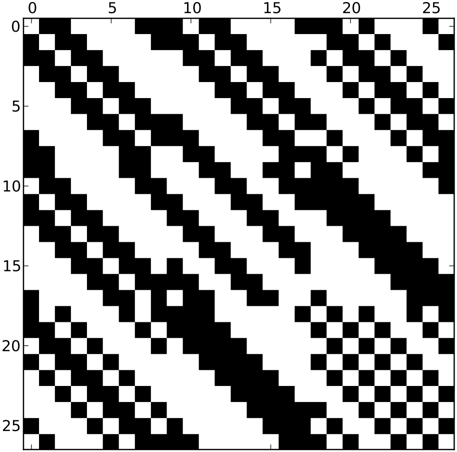


$9\#(1,3;2,1;3,2)$

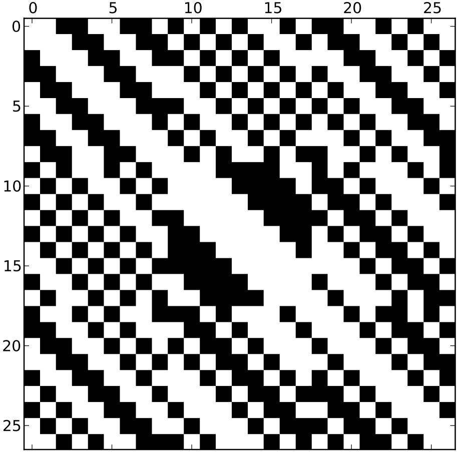


A, adjacency matrix

$\boxed{A = P^T B P}$ is the question



A



B

Isomorphism via SVD

Given A, B, seek permutation matrix P: $A = P^T BP$.

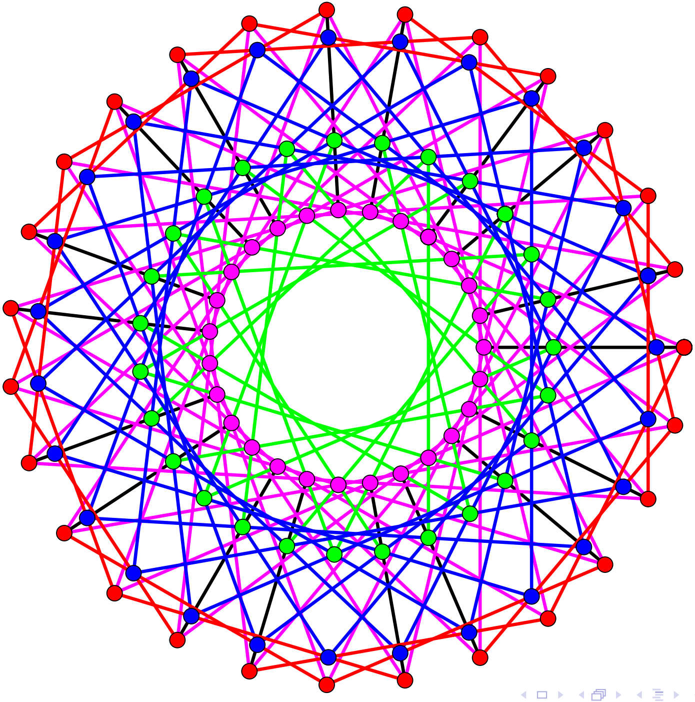
- compute svds, $A = U\Sigma V^T$, $B = \tilde{U}\tilde{\Sigma}\tilde{V}^T$
  $\longrightarrow$ negative result if $\Sigma \neq \tilde{\Sigma}$
- $A = (P^T \tilde{U})\tilde{\Sigma}(\tilde{P}^T V)^T$ is also a svd
- by uniqueness*, solve $P^T \tilde{U} = U$ instead of $A = P^T BP$
  $\longrightarrow$ success if we can distinguish rows of U

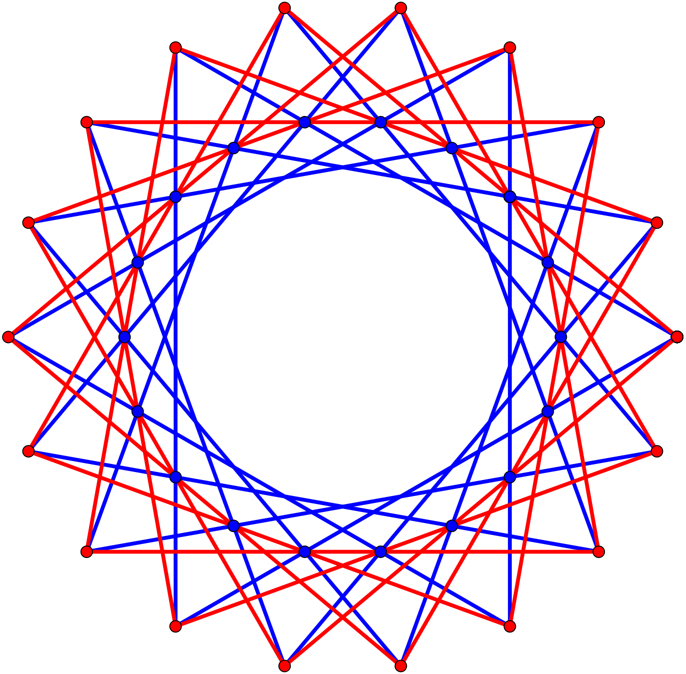
*: each u_j corresponding to an isolated σ_j is unique up to mult. by ± 1

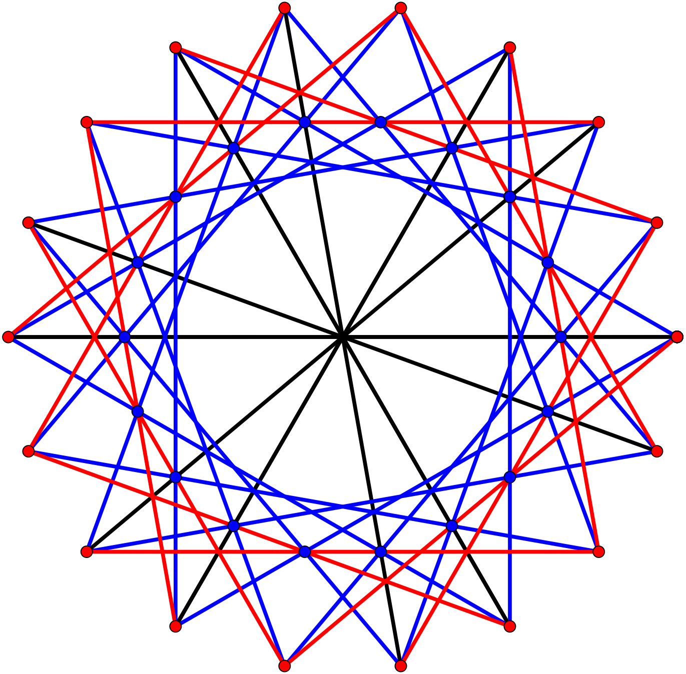
Conclusion

We have a numerical isomorphism test, but no theorem.

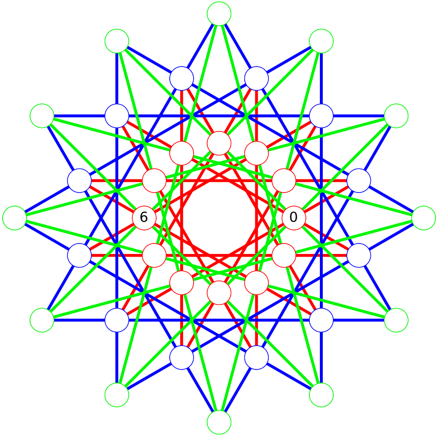
data
conjecture
proof



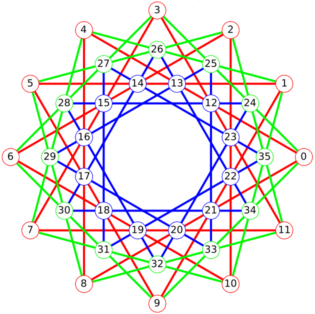




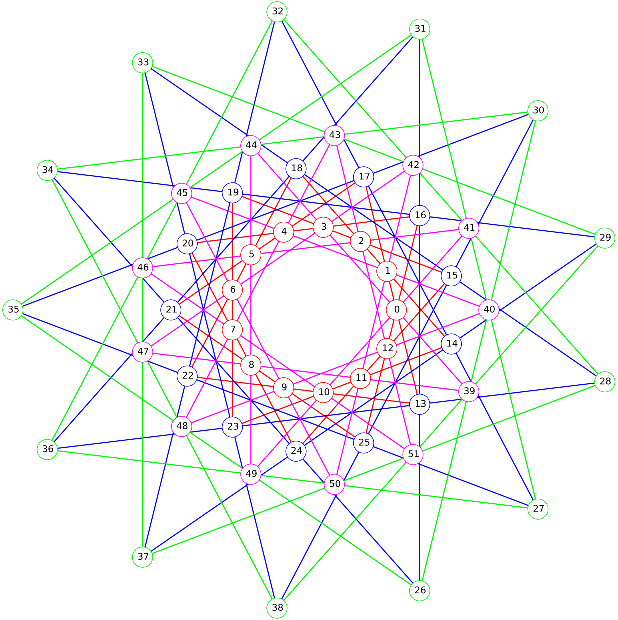
12#(4,5;3,4;5,3)



12#(4,1;3,4;1,3)



13#(1,4;3,5;4,1;5,3)



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13#(1,3;2,4;3,1;4,2)