

## Math 764. Homework 2

Due Friday, February 10th

### Extension of a sheaf by zero.

Let  $X$  be a topological space, let  $U \subset X$  be an open subset, and let  $\mathcal{F}$  be a sheaf of abelian groups on  $U$ .

1. The extension by zero  $j_!\mathcal{F}$  of  $\mathcal{F}$  (here  $j$  is the embedding  $U \hookrightarrow X$ ) is the sheaf on  $X$  that can be defined as the sheafification of the presheaf  $\mathcal{G}$  such that

$$\mathcal{G}(V) = \begin{cases} \mathcal{F}(V), & V \subset U \\ 0, & V \not\subset U. \end{cases}$$

Is the sheafification necessary in this definition? (Or maybe  $\mathcal{G}$  is a sheaf automatically?)

2. Describe the stalks of  $j_!\mathcal{F}$  over all points of  $X$  and the espace étalé of  $j_!\mathcal{F}$ .
3. Verify that  $j_!$  is the left adjoint of the restriction functor from  $X$  to  $U$ : that is, for any sheaf  $\mathcal{G}$  on  $X$ , there exists a natural isomorphism

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}|_U) \simeq \mathrm{Hom}(j_!\mathcal{F}, \mathcal{G}).$$

(The restriction  $\mathcal{G}|_U$  of a sheaf  $\mathcal{G}$  from  $X$  to an open set  $U$  is defined by  $\mathcal{G}|_U(V) = \mathcal{G}(V)$  for  $V \subset U$ .)

*Side question (not part of the homework):* What changes if we consider the version of extension by zero for sheaves of sets ('the extension by empty set')?

### Examples of affine schemes.

4. Let  $R_\alpha$  be a possibly infinite collection of rings. Put  $R = \prod_\alpha R_\alpha$ . Describe the topological space  $\mathrm{Spec}(R)$  in terms of  $\mathrm{Spec}(R_\alpha)$ 's.
5. Recall that the image of a regular map of varieties is constructible (Chevalley's Theorem); that is, it is a union of locally closed sets. Give an example of a map of rings  $R \rightarrow S$  such that the image of a map  $\mathrm{Spec}(S) \rightarrow \mathrm{Spec}(R)$  is
  - (a) An infinite intersection of open sets, but not constructible.
  - (b) An infinite union of closed sets, but not constructible.

### Contraction of a subvariety.

Let  $X$  be a variety (over an algebraically closed field  $k$ ) and let  $Y \subset X$  be a closed subvariety. Our goal is to construct a  $k$ -ringed space  $Z = (Z, \mathcal{O}_Z) = X/Y$  that is in some sense the result of 'gluing' together the points of  $Y$ . While  $Z$  can be described by a universal property, we prefer an explicit construction:

- The topological space  $Z$  is the 'quotient-space'  $X/Y$ : as a set,  $Z = (X - Y) \sqcup \{z\}$ ; a subset  $U \subset Z$  is open if and only if  $\pi^{-1}(U) \subset X$  is open. Here the natural projection  $\pi : X \rightarrow Z$  is identity on  $X - Y$  and sends all of  $Y$  to the 'center'  $z \in Z$ .
- The structure sheaf  $\mathcal{O}_Z$  is defined as follows: for any open subset  $U \subset Z$ ,  $\mathcal{O}_Z(U)$  is the algebra of functions  $g : U \rightarrow k$  such that the composition  $g \circ \pi$  is a regular function  $\pi^{-1}(U) \rightarrow k$  that is constant along  $Y$ . (The last condition is imposed only if  $z \in U$ , in which case  $Y \subset \pi^{-1}(U)$ .)

In each of the following examples, determine whether the quotient  $X/Y$  is an algebraic variety; if it is, describe it explicitly.

**6.**  $X = \mathbb{P}^2$ ,  $Y = \mathbb{P}^1$  (embedded as a line in  $X$ ).

**7.**  $X = \{(s_0, s_1; t_0 : t_1) \in \mathbb{A}^2 \times \mathbb{P}^1 : s_0 t_1 = s_1 t_0\}$ ,  $Y = \{(s_0, s_1; t_0 : t_1) \in X : s_0 = s_1 = 0\}$ .

**8.**  $X = \mathbb{A}^2$ ,  $Y$  is a two-point set (if you want a more challenging version, let  $Y \subset \mathbb{A}^2$  be any finite set).