A GENTLE INTRODUCTION TO DESCENT

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1. GAGS Talk 1 – April 26

1.1. "Concrete" examples.

1.1.1. Let $X = \operatorname{Spec} R$ be an affine scheme and $\{U_i \to X\}$ a family of morphisms of affine schemes corresponding to the ring homomorphisms $\{R \to R_i\}$. Then any R-module M, thought of as a module over X, determines modules M_i over each U_i by extension of scalars $M_i = R_i \otimes_R M$.

The modules M_i are **"restrictions"** or **pullbacks** of M along $U_i \to X$, and are **compatible** in that if $M_{i,j}$ is the pullback of M_i over U_i along $U_i \times_X U_j \to U_i$, then we have isomorphisms $M_{i,j} \cong M_{j,i}$ that satisfy certain coherence conditions that we'll describe later.

We say that modules over X have the property of **descent** for the family $\{U_i \to X\}$ if any family of compatible modules M_i over U_i glues uniquely to a module M over X that the M_i are restrictions of.

Example 1.1.2. Do modules have descent for a Zariski principal open cover $\{U_i \to X\}$, i.e. a family corresponding to prinicpal localizations $\{R \twoheadrightarrow R_{f_i}\}$ so that $1 \in \langle f_i \rangle$?

Yes: the partition-of-unity argument from a first course on algebraic geometry shows that modules have descent with respect to Zariski principal open covers.

Example 1.1.3. Do modules have descent for a Zariski cover $\{U \to X\}$ corresponding to not necessarily principal localizations $\{R \twoheadrightarrow S_i^{-1}R\}$?

Yes, as long as the cover has a finite subcover. The proof is word-for-word the same as the partition of unity argument for Zariski covers.

This kind of descent is useful, e.g. in studying curves with finitely many singularities. For such a singular curve, this form of descent implies that to specify a quasi-coherent sheaf on the curve, one need only specify a quasi-coherent sheaf on the smooth locus, modules over the stalks of the singularities, and isomorphisms between their restrictions to the generic point(s).

Example 1.1.4. Do modules have descent for a Zariski cover $\{U \to X\}$ corresponding to flat ring homomorphisms $\{R \to R_i\}$?

Yes, as long as the cover has a finite subcover.

This kind of descent is the affine scheme version of **fpqc-descent**; fpqc covers of schemes are the natural extension of these finite flat covers of affine schemes to all schemes.

1.1.5. Descent theory is a categorical formalization of the notion of descent. It relies on the notion of fibrations which we'll explore in accordance with the following plan.

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(1) "Concrete" examples
(2) Formalizing restrictions:
   - "Pseudofunctors" $\xleftrightarrow{\text{Grothendieck construction}}$ (cloven) fibrations
(3) Formalizing compatibility:
   - Representable fibrations
   - Sieves (subfibrations of representable fibrations)
   - Morphisms of fibrations and th fibrational Yoneda lemma
(4) Stacks and descent conditions:
   - Fibrations are analogous to presheaves
   - Prestacks are fibrations with uniqueness of gluing descent condition (analogous to separated presheaves)
   - Stacks are fibrations with unique gluing descent condition (analogous to sheaves)

This development is mostly distilled from Part I of FGA Explained.

1.2. Formalizing restrictions: pseudofunctors, the Grothendieck construction, fibrations.

Definition 1.2.1. Given a category $\mathcal{C}$, a **"pseudofunctor"** $\mathcal{F}$ associates

(1) To each object $X \in \mathcal{C}$ a category $\mathcal{F}(X)$
(2) To each morphism $X \xrightarrow{f} Y \in \mathcal{C}$ a **pullback functor** $\mathcal{F}(X) \xleftarrow{f^*} \mathcal{F}(Y)$.
(3) For each object X, a natural isomorphism $\mathrm{id}_X^* \cong \mathrm{id}_{\mathcal{F}(X)}$
(4) To each pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ a natural isomorphism $(g \circ f)^* \cong f^* g^*$

satisfying the **coherence axioms** that the following composites are the same:

- $f^* \mathrm{id}_X^* \cong f^* \mathrm{id}_{\mathcal{F}(X)}$ and $f^* \mathrm{id}_X^* \cong (\mathrm{id}_X \circ f)^*$
- $\mathrm{id}_X^* g^* \cong \mathrm{id}_{\mathcal{F}(X)} g^*$ and $\mathrm{id}_X^* g^* \cong (g \circ \mathrm{id}_X)^*$
- $f^* g^* h^* \cong f^*(h \circ g)^* \cong (h \circ g \circ f)^*$ and $f^* g^* h^* \cong (g \circ f)^* h^* \cong (h \circ g \circ f)^*$ are the same.

Example 1.2.2. For $\mathcal{C}$ the category of affine scheme, we have a "pseudofunctor" associating to each affine scheme $X = \operatorname{Spec} R$ the category of R-modules, and to each morphism $X' \to X$ corresponding to a ring homomorphism $R \to R'$, the extension-of-scalars functors $R'\text{-}\mathcal{M}od \xleftarrow{-\otimes_R R'} R\text{-}\mathcal{M}od$. The natural isomorphisms between composites of pullback functors are the natural isomorphisms between itereated tensor products; in particular, their coherence follows from the coherence of the natural isomorphisms of iterated tensor products.

Remark 1.2.3. "Pseudofunctor" is in quotes because its values can be large categories (like the category of R-modules), and there is no category of all large categories unless you adopt a stronger foundation such as Grothendieck universes. Nevertheless, even without Grothendieck universes, "pseudofunctor" is just as well-defined a notion as the notion of a large category, but you have to dig into first-order logic to understand how.

The "pseudo" in "pseudofunctor" refers to the fact that the association of morphisms to pullback functors is only functorial up to an explicit choice of natural isomorphisms; we would have merely a "functor" in the case where the natural isomorphisms are all identities.

Definition 1.2.4 (Grothendieck construction)**.** Given a "pseudofunctor" $\mathcal{F}$, we construct a category (abusively also labeled $\mathcal{F}$) and a functor $\mathcal{C} \xleftarrow{p} \mathcal{F}$ as follows.

(1) The collection of objects of $\mathcal{F}$ is the disjoint union of the collections of objects of $\mathcal{F}(X)$ for each $X \in \mathcal{C}$; the functor $\mathcal{C} \xleftarrow{p} \mathcal{F}$ sends an object $A \in \mathcal{F}(X)$ to $X \in \mathcal{C}$.

(2) A morphism $A \xrightarrow{\alpha} B \in \mathcal{F}$ from $A \in \mathcal{F}(X)$ to $B \in \mathcal{F}(Y)$ consists of a pair of morphisms $X \xrightarrow{f} Y \in \mathcal{C}$ and $A \xrightarrow{a} f^*B \in \mathcal{F}(X)$; the functor $\mathcal{C} \xleftarrow{p} \mathcal{F}$ sends a morphism $A \xrightarrow{(f,a)} B$ to $X \xrightarrow{f} Y$.

(3) The composite $A \xrightarrow{(f,a)} B \xrightarrow{(g,b)} C$ is given by $A \xrightarrow{(g\circ f,c)} C$ where the vertical morphism $A \xrightarrow{c} (g \circ f)^*C$ is given by the composite $A \xrightarrow{a} f^*B \xrightarrow{f^*b} f^*g^*C \cong (g \circ f)^*C$.

Exercise 1.2.5. Check that the Grothendieck construction does indeed produce a category. In other words, verify that composition is associative and that the pair $(X \xrightarrow{\mathrm{id}_X} X, A \cong \mathrm{id}_X^* A)$ is an identity morphism for $A \in \mathcal{F}$; you will have to use the coherence axioms satisfied by the natural isomorphisms of the "pseudofunctor".

Remark 1.2.6. The vertical fiber $p^{-1}(X)$ of objects mapping to X and morphisms mapping to $X \xrightarrow{\mathrm{id}_X} X$ is isomorphic to the category $\mathcal{F}(X)$. The **total category** $\mathcal{F}$ associated to a "pseudofunctor" has "vertical" morphisms that are the same as the values of the "pseudofunctor" and "horizontal" morphisms are the fewest morphisms that could come from morphisms in $\mathcal{C}$ in a sense we make precise below.

Definition 1.2.7. A functor $\mathcal{C} \xleftarrow{p} \mathcal{F}$ is a **fibration** if for every morphism $X \xrightarrow{f} Y$ and every object $B \in p^{-1}(Y)$, there is a morphism $A \xrightarrow{\alpha} B$ in $\mathcal{F}$ such that

(1) α is a **lift** of $X \xrightarrow{f} Y$ in the sense that $p\alpha = f$.

(2) $A \xrightarrow{\alpha} B$ is p**-cartesian** in the sense that whenever the projection $pA' \xrightarrow{p\beta} pB = Y$ of a morphism $A' \xrightarrow{\beta} B$ factors as $pA' \xrightarrow{g} pA \xrightarrow{p\alpha} pB$, $A' \xrightarrow{\beta} B$ factors uniquely as $A' \xrightarrow{\gamma} A \xrightarrow{\alpha} B$ with $p\gamma = g$.

A **cloven fibration** is a fibration with a **cleavage**, that is, a choice of p-cartesian lifts $f^*B \to B \in \mathcal{F}$ for each pair of a morphism $X \xrightarrow{f} Y \in \mathcal{C}$ and an object $B \in p^{-1}(Y)$.

Remark 1.2.8. As we describe below, the Grothendieck construction produces a cloven fibration out of a pseudofunctor, and inversely every cleavage of a fibration determines a "pseudofunctor". The advantage of fibrations over "pseudofunctors" is that the structure of natural isomorphisms between pullback functors is automatically induced and kept track of by the universal properties of p-cartesian morphisms. Consequently, the theory of fibrations is simpler to describe and develop than the corresponding theory of "pseudofunctors".

Example 1.2.9. Consider the codomain functor $\mathcal{C} \xleftarrow{\mathrm{cod}} \mathcal{C}^{\to}$ be the category whose objecsts are morphisms $A \to X \in \mathcal{C}$, and whose morphisms are commutative squares. A lift of a morphism $X \xrightarrow{f} Y \in \mathcal{C}$ along an object $Y \xleftarrow{b} B$ in $\mathrm{cod}^{-1}(Y)$ consists of a pair of morphisms $X \xleftarrow{a} A \xrightarrow{\phi} B$ forming a commutative square with the pair $X \xrightarrow{f} Y \xleftarrow{b} B$.

This lift will be cod-cartesian if whenever we have a pair of morphisms $X' \xleftarrow{a'} A' \xrightarrow{\beta} A$ that form a commutative square with $X' \xrightarrow{g} X \xrightarrow{f} Y \xleftarrow{b} B$, there is a unique morhism $A' \xrightarrow{\gamma} A$ so that $A' \xrightarrow{\beta} A$ factors as $A' \xrightarrow{\beta} A \xrightarrow{\alpha} B$ and the square formed by $X' \xleftarrow{a'} A' \xrightarrow{\gamma} A$ and $X' \xrightarrow{g} X \xleftarrow{a} A$ commutes.

In other words, cod-cartesian morphisms are exactly pullback squares.

Exercise 1.2.10. The pullback lemma in basic category theory states that when two commutative squares are pasted together and the right-hand commutative square is a pullback square, then the left-hand square is a pullback square if and only if the whole rectangle is a pullback.

Verify that this is a general propery of p-cartesian morphisms for any functor $\mathcal{C} \xleftarrow{p} \mathcal{F}$, i.e. if $B \xrightarrow{\beta} C$ is p-cartesian, then $A \xrightarrow{\alpha} B$ is p-cartesian if and only if the composite $\beta \circ \alpha$ is p-cartesian.

Exercise 1.2.11. The functor $\mathcal{C} \xleftarrow{p} \mathcal{F}$ produced by the Grothendieck construction is a fibration with clevage given by $f^*B \xrightarrow{(f,\mathrm{id}_{f^*B})} B$ for any $B \in p^{-1}(Y)$ and $X \xrightarrow{f} Y$.

Reversely, given a morphism $X \xrightarrow{f} Y$, a choice of p-cartesian lifts $f^*B \to B$ for each $B \in p^{-1}(Y)$ extends by the universal propery of p-cartesian morphisms to a pullback functor $p^{-1}(X) \xleftarrow{f^*} p^{-1}(Y)$. Furthermore, and again by the universal property of p-cartesian morphisms, a cleavage for a fibration also determines the requisite coherent natural isomorphisms between composites of the pullback functors just described.

1.3. Formalizing compatibility: representable fibrations.

Example 1.3.1. Consider the domain functor $\mathcal{C} \xleftarrow{\mathrm{dom}} \mathcal{C}^{\to}$. Given a morphism $X \xrightarrow{f} Y$ and an object $Y \xrightarrow{b} B$ in $\mathrm{dom}^{-1}(Y)$, a lift is a pair of morphism $X \xrightarrow{a} A \xrightarrow{\alpha} B$ so that the together with $X \xrightarrow{f} Y \xrightarrow{b} B$ they form a commutative square.

This lift is dom-cartesian if for any pair of morphisms $X \xrightarrow{a'} A' \xrightarrow{\beta} B$ determing a commutative square with $X' \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{b} B$, there is a unique morphism $A' \xrightarrow{\gamma} A$ through which $A' \xrightarrow{\beta} B$ factors as $A' \xrightarrow{\gamma} A \xrightarrow{\alpha} B$, and such that the square determined by $X' \xrightarrow{a'} A' \xrightarrow{\gamma} A$ and $X' \xrightarrow{g} X \xrightarrow{a} A$ commutes.

Evidently, $X \xrightarrow{b \circ f} B \xrightarrow{\mathrm{id}_B} B$ determines a dom-cartesian lift. In particular, $\mathcal{C} \xleftarrow{\mathrm{dom}} \mathcal{C}^{\to}$ is always a fibration, and a pullback functor associated to $X \xrightarrow{f} Y$ is the pre-composition functor $X/\mathcal{C} \xleftarrow{f^*} Y/\mathcal{C}$ (where $X/\mathcal{C} = \mathrm{dom}^{-1}(X)$ is the category whose objects are morphisms out of X and whose morphisms are commutative triangles) .

Remark 1.3.2. The particular choice of pre-composition lifts of the domain fibration is such that the natural isomorphisms of the associated "pseudofunctor" are all identities. Such a cleavage of a fibration is called a **splitting**, and the Grothendieck construction actually establishes a correspondence between split fibrations and "functors".

Definition 1.3.3. For each object X, the functor $\mathcal{C} \xleftarrow{\mathrm{dom}} \mathcal{C}/X$ is called a **representable fibration** (where $\mathcal{C}/X$ is the category whose objects are morphisms into X and whose morphisms are commutative triangles). Its standard pullback functors are also given by pre-composition, meaning that $\mathcal{C} \xleftarrow{\text{dom}} \mathcal{C}/X \hookrightarrow \mathcal{C}^{\to}$ is a **subfibration** of $\mathcal{C} \xleftarrow{\text{dom}} \mathcal{C}^{\to}$.

Remark 1.3.4. A reason for the name "representable fibrations" is that on objects, the fibration $\mathcal{C} \xleftarrow{\text{dom}} \mathcal{C}/X$ with its standard splitting determines exactly the "functor" $\mathrm{Hom}_{\mathcal{C}}(-, X)$ (with "functor" in quotes because the category $\mathcal{C}$ may be large, i.e. the values $\mathrm{Hom}_{\mathcal{C}}(Y, X)$) may be a class rather than a set).