LINEAR ALGEBRA (02/09/22)

Putnam club webpage: `https://www.math.wisc.edu/wiki/index.php/Putnam_Club`
To signup for the mailing list/Google groups, email `putnam-club+join@g-groups.wisc.edu`

Easier problems

1. The polynomial $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ has integer coefficients and n distinct integer roots. Suppose that all of its roots are coprime. Show that a_n and a_{n-1} are coprime.

2. a, b, and c are the three roots of the polynomial $x^3 - 3x^2 + 1$. Find $a^3 + b^3 + c^3$.

3. $p(x)$ is a polynomial with integer coefficients such that $p(0) = 1$, $p(1) = 2$, $p(-1) = 4$. Prove that $p(x)$ has no integer roots. (In the first version of this problem, I had $p(-1) = 1$. Why did I change it?)

4. (AIME 1989) Assume that $x_1, x_2, \ldots, x_7$ are real numbers such that

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1 \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12 \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123. \end{aligned}$$

Find the value of

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7.$$

Putnam problems (not necessarily harder)

5. (Putnam 2009) (From last time) Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, ..., \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

(The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

6. (Putnam 2009) Let $p(x)$ be a real polynomial that is nonnegative for all real x. Prove that for some k, there are real polynomials $f_1(x), \ldots, f_k(x)$ such that $p(x) = \sum_{i=1}^{k} f_i(x)^2$.

7. (Putnam 2003) Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)?$$

8. (Putnam 2010) Find all polynomials $P(x)$, $Q(x)$ with real coefficients such that $P(x)Q(x+1) - P(x+1)Q(x) = 1$.

9. (Putnam 2008) Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points

$$(f(1), g(1)), (f(2), g(2)), \ldots, (f(n), g(n)) \in \mathbb{R}^2$$

are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater or equal to $n-1$. (By the way, suppose n is odd and $\deg(f(x)) < n-1$. Can you say anything special about the regular n-gon?)

10. (Putnam 2011) For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

Hints