Putnam Club
November 13, 2024
Number Theory

1. There are infinitely many composite numbers in the sequence 1, 31, 331, 3331, etc.

2. (Virginia Tech Regional Competition, 1988) Let a be a positive integer. Find all positive integers n such that $b = a^n$ satisfies the condition that $a^2 + b^2$ is divisible by $ab + 1$

3. Let p be a positive integer such that the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{p}$$

has exactly 3 solutions in positive integers. Show that p is prime. (Solutions (x, y) and (y, x) are considered distinct.)

4. (Putnam 2018-A1). Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

Harder:

5. Let p, q be relatively prime positive integers. Prove that

$$\sum_{k=0}^{pq-1} (-1)^{\lfloor \frac{k}{p} \rfloor + \lfloor \frac{k}{q} \rfloor} = \begin{cases} 0 \text{ if } pq \text{ is even,} \\ 1 \text{ if } pq \text{ is odd.} \end{cases}$$

6. (Putnam 2014-B3). Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least $m + n$ distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.

7. (Putnam 2001-B4). Let S denote the set of rational numbers different from -1, 0 and 1. Define $f : S \to S$ by

$$f(x) = x - (1/x).$$

Prove or disprove that

$$\bigcup_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)} = f \circ f \circ \cdots \circ f$ (n times).

Hard:

8. (Putnam 1998-B5). Let N be the positive integer with 1998 decimal digits, all of them 1; that is, $N = 1111\ldots11$ (1998 digits). Find the thousandth digit after the decimal point of $\sqrt{N}$.

9. (Putnam 1981-B5). Let $B(n)$ be the number of ones in the base two expression for the positive integer n. For example, $B(6) = B(110_2) = 2$ and $B(15) = B(1111_2) = 4$. Determine whether or not

$$\exp\left(\sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)}\right)$$

is a rational number. Here $\exp(x)$ denotes e^x.

10. (Putnam 2012-B6). Let p be an odd prime such that $p \equiv 2 \mod 3$. Define a permutation π of the residue classes modulo p by $\pi(x) \equiv x^3 \mod p$. Show that π is an even permutation if and only if $p \equiv 3 \mod 4$.

Hints and comments:

Frankly, some of these are what I would call 'non-standard number theory': they are problems about numbers, but do not use any deep number theory facts. Well, sometimes you get bored of all the congruence/divisibility/Euler's Theorem questions...

Here are hints if you need them:

1. Choose a prime and show that the remainders modulo this prime will be periodic.
2. What happens when you divide $a^m + 1$ by $a^k + 1$?
3. Bring everything to common denominator and simplify.
4. Same.
5. At least one of these should be easy.
6. What happens if A has rank 1?
7. Look at the reduced fraction form of x and $f(x)$.
8. N is very close to a square of a rational number.
9. Separate the sum into terms coming from one in each particular position.
10. I think you need two things: existence of a primitive root modulo p and the quadratic reciprocity. Look these up if you need to!