NINTH ANNUAL UW MADISON UNDERGRADUATE MATH COMPETITION

1. Suppose f is a differentiable function. Show that there exists α between 0 and 2π such that the vector $(f(\alpha), f'(\alpha)+1)$ is perpendicular to the unit vector $(\cos(\alpha), \sin(\alpha))$.

Solution: Put $g(\alpha) = -\cos(\alpha) + f(\alpha)\sin(\alpha)$. Then $g(0) = g(2\pi) = -1$, and by the Mean Value Theorem, there is $\alpha \in (0, 2\pi)$ such that $g'(\alpha) = 0$. It remains to notice that $g'(\alpha)$ is the dot product of the given two vectors.

2. Find all pairs of integers $a, p > 1$ such that p is prime and $\log_a(a+p)$ is rational (where $\log_a$ is the logarithm base a).

Solution: The only such pair is $(a, p) = (2, 2)$. Indeed, suppose the logarithm is n/m. We then have

$$a^n = (a+p)^m.$$

Note that $n, m > 0$.

If a and $a + p$ are coprime, this is impossible. The only possible common factor of a and $a+p$ is p, and we then have $a = bp$:

$$b^n p^n = (b+1)^m p^m.$$

Note that if b is divisible by some prime other than p, then this prime appears on the left, but not on the right of the equation, because b and $b+1$ are coprime, so b must be a power of p. Similarly, $b+1$ must be a power of p. This implies $b = 1$, $b+1 = 2$, and $p = 2$, as claimed.

3. Let A be a 100×100 matrix such that each column of A contains at most two entries 1, and the rest of entries are zero. What is the maximal value of $\det(A)$?

Solution: Let us try to maximize $|\det(A)|$ (we can always switch two rows instead).

The answer is $2^3 3$. If some column of A contains no 1's, then $\det(A) = 0$. If some column contains exactly one 1, we can remove this column (and the corresponding row), and reduce the size of matrix by 1. Similar argument works if some row contains exactly one 1.

It remains to consider the case when the matrix has exactly two 1's in each row and in each column. Permuting rows and columns, we can bring the matrix to block-diagonal form, with all blocks of the form

$$\begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \dots & & & & \\ 1 & 0 & 0 & \dots & 1 \end{pmatrix},$$

whose determinant is either 2 or zero, depending on whether the size of the block is odd or even. This shows that we maximize the determinant by using as many 3×3 blocks as possible, which gives the answer.

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4. In each square of a 2025×2025 board there is a light and a switch. Flicking a switch changes the state of the light (on to off and off to on) in its square, as well as all the lights in the same row and in the same column. In the beginning, all lights are off. How many different configurations of lights can be obtained by using the switches?

Solution: The answer is $2^{2025^2-2\times 2024}$. The set of all positions of switches form a vector space over $(\mathbb{Z}/2\mathbb{Z})$ of dimension 2025^2, and the map sending the position of switches to the resulting configuration of lights is linear. Let us find the kernel of this map, that is, the set of positions of switches that result in all lights being off.

Denote by $C \subset \{1, \ldots, 2025\}$ the set of columns with *odd* number of switches that are on, and by $C \subset \{1, \ldots, 2025\}$ the set of rows with the same property. In order for the light in the position (i, j) to be off, the switch must be off exactly when either $i \in C$, $j \in R$, or $i \notin C$, $j \notin R$. Thus, in i-th column, there are exactly $2025 - |R|$ switches on if $i \in C$, and $|R|$ if $i \notin C$; in either case, this implies that $|R|$ is even. Similarly, $|C|$ is even. The number of ways to choose a subset of even cardinality from $\{1, \ldots, 2025\}$ is 2^2024, so the total number of ways to use switches so that all the lights are off is $2^{2\times 2024}$

5. Let $f(x) > 0$ be a function defined for all x and suppose M is a constant such that $f''(x) \leq M$ for all x (in particular, $f(x)$ is two times differentiable.) Show that

$$(\sqrt{f})' \leq \sqrt{M/2}$$

Solution: Suppose $f(x_0) = a$ and $f'(x_0) = b$; by the hypothesis, $a \geq 0$. Then it is easy to see that

$$f(x) \leq a + b(x - x_0) + (M/2)(x - x_0)^2$$

(for instance, by the Taylor formula). In order for f to be positive, the right-hand side must be always positive, which means its discriminant must be negative:

$$b^2 < aM.$$

But then $(\sqrt{f(x)})' = b/2\sqrt{a} < \frac{1}{2}\sqrt{M}$. (The weaker estimate given in the problem comes from a different solution.)

6. Let n be an integer. Consider a random sequence of sets $B_0, B_1, \ldots, B_n$ chosen recursively so that $B_0 = \{1, \ldots, 3^n\}$, and B_{k+1} is a randomly chosen subset of B_k for $k \geq 0$, with all choices equally likely. Denote by P_n the probability that $B_n = \emptyset$. Find

$$\lim_{n\to\infty} P_n.$$

Solution: For a random choice of a subset of a set, each element has a $1/2$ probability of belonging to the subset. Thus, each element of B_0 has probability $1/2$ to be in B_1, $1/4$ to be in B_2, etc, and probability 2^{-n} to be in B_n. Moreover, all choices are independent, so the probability that B_n is empty is

$$(1 - 2^{-n})^{3^n}.$$

Since $\lim(1 - 2^{-n})^{2^n} = e^{-1}$, we see that the limit is zero. (Maybe it would have been more logical to have $B_0 = \{0, \ldots, 2^n\}$, but I decided to vary it a little.)